Intertemporal Risk Management in Agriculture

Scott Colby[1], Timothy Graciano[1], Jeffrey LaFrance[1,2,3], Rulon Pope[4], and Jesse Tack[5]

Abstract: Agricultural production is subject to supply risk. Expected and realized farm outputs and output prices are unknown and unobservable when inputs are chosen. Crop and livestock production decisions are linked over time. Producers' expectations are particularly difficult to model. This paper presents the necessary and sufficient condition to allow input demands to be specified as functions of input prices, technology, quasi-fixed inputs, and cost in place of planned/expected outputs. These all are observable when inputs are committed to production. Next we derive a flexible, exactly aggregable, economically regular econometric model of input demands. This model is consistent with any dynamic von Newman – Morgenstern expected utility function. We combine this framework with a model of the life-cycle production, investment and savings, and consumption decisions of owner/operators who face output and output price risk, and who have opportunities to invest in a conditionally risk free asset, other risky financial assets, and farm assets. The econometric framework allows for location specific technological change and production processes, cross-equation, interspatial, and intertemporal correlation among the error terms, and structural simultaneity between inputs and outputs, input and output prices, investment in durable goods used in agriculture, consumption, savings, and wealth. The result is a consistent dynamic structural model of inputs, outputs, savings, investment, and consumption under risk. Ongoing empirical work applies this model at the state-level to crop and livestock production for the years 1960-2004. An additional ongoing effort is to update this data set to 2008.

Key Words: Aggregation, consumption, ex ante cost, expected utility, functional form, investment, life cycle, rank, risk, savings

JEL Classification: C3, D2, D8

Please address correspondence to:

Professor Jeffrey T. LaFrance
School of Economic Sciences
101 Hulbert Hall
PO Box 646210
Washington State University
Pullman, WA 99164-6210
jtlafrance@wsu.edu

[1] School of Economic Sciences and Paul Allen School for Global Animal Health, Washington State University
[2] Department of Agricultural and Resource Economics, University of California–Berkeley
[3] Department of Economics, Monash University
[4] Department of Economics, Brigham Young University
[5] Department of Agricultural Economics, Mississippi State University

1. Introduction

Farm and food policies affect crop acres, asset management, intensive and extensive margin decisions, and risk management choices in agricultural production. For example, in 1991, less than 25% of cropland (82 million acres) was covered by a Federally subsidized crop insurance contract, with $11.2 billion in total liability, $740 million in insurance premiums, premium subsidies of 25% ($190 million) of gross farm premiums, and total indemnity payments of $955 million. Relative to premiums paid by farmers ($550 million), for each $1.00 in premiums paid by the typical insured farmer, $1.75 in indemnity payments were received.

Even with this relatively profitable insurance program, farmer participation rates remained quite low. This outcome is likely due to the race to the bottom problem in a pooling equilibrum (LaFrance, Shimshack, and Wu 2000, 2001, 2002, 2004). However, Congress responded to the *appearance* of an incomplete insurance market with increased subsidies and many new forms of insurance.

The 1996 Federal Agricultural Improvement and Reform Act and the amendments to the 1938 Federal Crop Insurance Act that are commonly known as the Agricultural Risk Protection Act of 2000 mandated higher subsidy rates, the development and marketing of new insurance products for virtually every crop and livestock product produced in the U.S., and substantial subsidies for crop insurance marketing firms and large private reinsurance companies.

This change in farm policy greatly expanded the Federal crop insurance program. In 2003, the Federal Crop Insurance Corporation (FCIC) provided insurance products for more than 100 crops on 217 million acres (2/3 of all cropland). The total insurance liability was $40.6 billion, with $3.4 billion in insurance premiums, subsidies of almost 60% of gross premiums ($2.0 billion), and total indemnity payments of $3.2 billion. The current program includes subsidy payments to private companies marketing Federal crop insurance equal to 24.5% of gross premiums for administration and oversight (A&O), and to private reinsurance companies equal to 13.6% of gross premiums. Reinsurance companies also have the right to sell up to 50% of their contracts back to the FCIC (that is, to the taxpayer) at cost. The FCIC's Risk Management Agency's (RMA) book of business shows that 20% of the insured farms account for nearly 80% of indemnity payments. This suggests substantial adverse selection, as well as moral hazard, since the majority of the Federally subsidized crop insurance products calculate premiums based on deviations from county-level yield trends. That is to say, FCIC insurance products are based on a pooling equilibrium established at the county level, and in some cases larger areas known as risk regions.

The net effect is that for each $1.00 in premiums actually paid by farmers they receive an average of $2.40 in indemnity payments, insurance marketing firms receive $0.40 in A&O subsidies, and reinsurance companies make in the neighborhood of $0.45 in profit due to the combined direct subsidies on premiums and their reinsurance rights with the FCIC, which allow them to cream, or high grade, the insurance pool.

In 2004 the RMA issued an RFP to develop subsidized pasture and range insurance for 440 million acres of private, public, and Native American pasture and rangeland in the country. Many agricultural economists at land grant universities across the country actively consult with the RMA and private insurance companies to develop new and expand existing Federally subsidized crop insurance products.

Although this is only one example of the ubiquitous nature of Federal intervention in U.S. agriculture, there is a large literature on the impacts of subsidized crop insurance on variable input use and the intensive margin (Nelson and Loehman 1987, Chambers 1989, and Quiggin 1992, Horowitz and Lichtenberg 1994, Smith and Goodwin 1996, and Babcock and Hennessy 1996). The effects of subsidized crop insurance programs on the extensive margin also has been the subject of considerable analysis (Gardner and Kramer 1986, Goodwin, Smith and Hammond 1999, Keeton, Skees and Long 1999, and Young, Schnepf, Skees, and Lin 1999, and LaFrance, Shimshack, and Wu 2000, 2001, 2002, 2004), all of which conclude that subsidized crop insurance results in additional of marginal crop acres. Williams (1988), Turvey (1992), Wu (1999), and Soule, Nimon, and Mullarkey (2000) examine the impacts of subsidized crop insurance on choices of crop mixes and acreage decisions. Empirical results in this component of the literature suggest that economically marginal land also is environmentally marginal. These results all suggest that subsidized crop insurance tends to increase environmental degradation. Even so, very little of the previous work in this area uses structural models, or takes into account the dynamic nature of agricultural decision making under risk.

To better understand these and many other longstanding issues in U.S. agricultural policy, this paper develops a comprehensive structural econometric model of variable input use, crop mix and acreage choices, investment and asset management decisions, and consumption, savings and wealth accumulation in a stochastic dynamic programming model of farm-level decision making over time. This model develops and establishes clear and intuitively appealing relationships between dynamic life-cycle consumption theory, the theory of the competitive firm subject to risk, and modern finance theory.

We present, discuss, and apply a new class of variable input demand systems in a multi-product production setting. All of the models in this class can be estimated with observable data, are exactly aggregable, are consistent with economic theory for any von Neumann-Morgenstern expected utility function, and can be used to nest and test exact aggregation, economic regularity, functional form, and flexibility. Implications of monotonicity, concavity in prices, and convexity in outputs and quasi-fixed inputs are developed for a specific subset of this class of models. We are currently in the process of applying this to 13 variable inputs in U.S. agriculture over the sample period 1960-2004 using state-level data.

This variable cost model is then used to help develop a structural model of the dynamic decision problems faced by a generic agricultural producer in each state. In this life-cycle model of agricultural decisions under risk, farmers create income and wealth through savings, investment in risky financial assets, own-labor choices both on- and off-farm, and agricultural production and investment activities. This disciplines the economic theory of agricultural production over time and under risk, and helps to better identify risk preferences and other model parameters.

2. The Production Model and Two Results

Four longstanding questions in economics, econometrics, and agricultural economics are the choice of functional form, the degree of flexibility, the conditions required for and regions of economic regularity, consistency with aggregation from micro- to macro-level data, and how best to handle simultaneous equations bias, errors in variables, and latent variables in a structural econometric models. In this paper, we attempt to deal with all of these issues in a coherent framework for the analysis of a life-cycle model of agricultural production, investment, consumption, and savings decisions.

Analysis of multi-product behavior of firms is common in economics (Färe and Primont 1995; Just, Zilberman, and Hochman 1988; Shumway 1983, Lopez 1983; Akridge and Hertel 1986). A large literature on functional structure and duality guides empirical formulations and testing based on concepts of non-jointness and separability (Lau 1972, 1978; Blackorby, Primont and Russell 1977, 1978; Chambers 1984). Non-joint production processes reduce to additivity in costs (Hall 1973; Kohli 1983). Separability in a partition of inputs or outputs often results in separability in a similar partition of prices (Blackorby, Primont and Russell 1977; Lau 1978).

The neoclassical model of conditional demands for variable inputs with joint production, quasi-fixed inputs, and production and output price risk is

$$\boldsymbol{x}(\boldsymbol{w},\bar{\boldsymbol{y}},\boldsymbol{z}) = \arg\min\left\{\boldsymbol{w}^{\top}\boldsymbol{x} : F(\boldsymbol{x},\bar{\boldsymbol{y}},\boldsymbol{z}) \leq 0\right\}, \tag{1}$$

where $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}_{+}^{n_x}$ is an n_x–vector of variable inputs, $\boldsymbol{w} \in \mathcal{W} \subseteq \mathbb{R}_{+}^{n_x}$ is an n_x–vector of variable input prices, $\bar{\boldsymbol{y}} \in \mathcal{Y} \subseteq \mathbb{R}_{+}^{n_y}$ is an n_y–vector of planned outputs, $\boldsymbol{z} \in \mathcal{Z} \subseteq \mathbb{R}_{+}^{n_k}$ is an n_z–vector of quasi-fixed inputs.[1] $F : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is the joint production transformation function, which is the boundary of a closed and convex production possibilities set that is characterized by free disposal in inputs and outputs. Let the variable cost function be denoted by $c(\boldsymbol{w},\bar{\boldsymbol{y}},\boldsymbol{z}) \equiv \boldsymbol{w}^{\top}\boldsymbol{x}(\boldsymbol{w},\bar{\boldsymbol{y}},\boldsymbol{z})$. We assume throughout that the production process is subject to supply shocks of the general form

[1] In this section, we use $\bar{\boldsymbol{y}} \in \mathbb{R}_{+}^{n_y}$ to denote the n_y–vector of planned/expected outputs to simplify notation. In later sections, we modify this notation to $\bar{\boldsymbol{Y}} = \boldsymbol{a} \bullet \bar{\boldsymbol{y}}$, where $\boldsymbol{a}$ is the n_y–vector of acres planted to crops, $\bar{\boldsymbol{y}}$ now is the n_y–vector of expected yields, and $\bullet$ is the Hadamard product. We also define $\boldsymbol{z}$ explicitly below.

$$\boldsymbol{y} = \overline{\boldsymbol{y}} + \boldsymbol{h}(\overline{\boldsymbol{y}}, \boldsymbol{z}, \boldsymbol{\varepsilon}),\; E\left[\boldsymbol{h}(\overline{\boldsymbol{y}}, \boldsymbol{z}, \boldsymbol{\varepsilon}) \mid \boldsymbol{x}, \overline{\boldsymbol{y}}, \boldsymbol{z}\right] = \boldsymbol{0}. \tag{2}$$

In either a static or a dynamic setting, it is a simple matter to show that (1) is implied by (2) and the expected utility hypothesis for *all* von Newman-Morgenstern preferences (Pope and Chavas 1994; Ball, et al., 2010).

Planned output is a vector of latent, unobservable variables in production with supply risk. Hence, to estimate the demand system in (1) directly, one must either identify and estimate the expectations formation process or address the errors in variables problem associated with using $\boldsymbol{y}$ in place of $\overline{\boldsymbol{y}}$ in the demand equations (Pope and Chavas 1994). One branch of the literature advocates specifying an *ex ante* cost function where planned output is replaced by cost, which is observable when the variable inputs are committed to the production process (Pope and Chavas 1994; Pope and Just 1998; Chambers and Quiggin 2000; Chavas 2008; Ball, et al. 2010; LaFrance and Pope 2010). In a joint production process, this requires making assumptions such that the input demands are functions of input prices, the levels of quasi-fixed inputs, and the variable cost of production,

$$\boldsymbol{x}(\boldsymbol{w}, \overline{\boldsymbol{y}}, \boldsymbol{z}) = \tilde{\boldsymbol{x}}(\boldsymbol{w}, \boldsymbol{z}, c(\boldsymbol{w}, \overline{\boldsymbol{y}}, \boldsymbol{z})). \tag{3}$$

This approach makes particular sense in agriculture where outputs and output prices are observed *ex post*. The main result of LaFrance and Pope (2010) on this question is as follows (a proof of this result is presented in Appendix A of this paper).

Proposition 1*: The following functional structures are equivalent:*

$$\boldsymbol{x}(\boldsymbol{w}, \overline{\boldsymbol{y}}, \boldsymbol{z}) \equiv \tilde{\boldsymbol{x}}(\boldsymbol{w}, c(\boldsymbol{w}, \overline{\boldsymbol{y}}, \boldsymbol{z}), \boldsymbol{z}); \tag{4}$$

$$c(\boldsymbol{w}, \overline{\boldsymbol{y}}, \boldsymbol{z}) \equiv \tilde{c}(\boldsymbol{w}, \boldsymbol{z}, \theta(\overline{\boldsymbol{y}}, \boldsymbol{z})); \tag{5}$$

$$F(\boldsymbol{x}, \overline{\boldsymbol{y}}, \boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x}, \boldsymbol{z}, \theta(\overline{\boldsymbol{y}}, \boldsymbol{z})). \tag{6}$$

In other words, outputs must be weakly separable from the variable input prices in the variable cost function. This, in turn, is equivalent to outputs being weakly separable from the variable inputs in the joint production transformation function.

This is a tight result – separability is both necessary and sufficient for the variable inputs to be estimable in *ex ante* form. Hereafter, we will call any such demand model an *ex ante joint production system.*

A second common issue in the empirical analysis of agricultural supply decisions is that some level of aggregation is virtually unavoidable. Micro-level data needed to study input use, acreage allocations, and asset management choices at the farm level does not exist. Aggregation from micro-level decision makers to macro-level data has been studied extensively in consumer theory.[2] This has received less attention in production economics (Chambers and Pope 1991, 1994; Ball et, al., 2010; LaFrance and Pope 2008, 2010).

Recently, LaFrance and Pope (2009) obtained the indirect preferences for all exactly aggregable, full rank systems of consumer demand equations. Their result extends directly to production in the following way. Let $K \in \{1,2,3,4\}$ and define the smooth real-valued function, $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, by

$$\omega(\eta(\boldsymbol{w}), \theta) = \begin{cases} \theta, & \text{if } K = 1,2, \text{ or } K = 3 \text{ and } \lambda'(s) = 0, \\ \theta + \int_0^{\eta(\boldsymbol{w})} \left[\lambda(s) + \omega(s,\theta)^2 \right] ds, & \text{if } K = 3,4, \text{ and } \lambda'(s) \neq 0, \end{cases} \tag{7}$$

subject to $\omega(0,\theta) = \theta$ and $\partial\omega(0,\theta)/\partial s = \lambda(0) + \theta^2$, where $\eta : \boldsymbol{\mathcal{W}} \to \mathbb{R}$ and $\lambda : \mathbb{R} \to \mathbb{R}$ are smooth, real-valued functions, and η is 0° homogeneous. A class of full rank and exactly aggregable *ex ante* production systems can be characterized as follows.[3]

Proposition 2: *Let* $\pi : \boldsymbol{\mathcal{W}} \to \mathbb{R}_{++}$, $\pi \in \mathcal{C}^\infty$, *be strictly positive valued, increasing, concave, and* 1° *homogeneous; let* $\eta : \boldsymbol{\mathcal{W}} \to \mathbb{R}_{+}$, $\eta \in \mathcal{C}^\infty$, *be positive valued and* 0° *homogeneous; let* $\alpha,\beta,\gamma,\delta : \boldsymbol{\mathcal{W}} \to \mathbb{C} = \{a + \imath b,\ a,b \in \mathbb{R}\}$, $\alpha,\beta,\gamma,\delta \in \mathcal{C}^\infty$, *be* 0° *homogeneous and satisfy* $\alpha\delta - \beta\gamma \equiv 1$, $\imath = \sqrt{-1}$; *and let* $f : \mathbb{R}_{++} \to \mathbb{C}$, $f \in \mathcal{C}^\infty$, *and* $f' \neq 0$. *Then the variable cost function for any full rank, exactly aggregable, ex ante joint production system is a special case of*

$$f\left(\frac{c(\boldsymbol{w}, \bar{\boldsymbol{y}}, \boldsymbol{z})}{\pi(\boldsymbol{w})}\right) = \frac{\alpha(\boldsymbol{w})\omega(\eta(\boldsymbol{w}), \theta(\bar{\boldsymbol{y}}, \boldsymbol{z})) + \beta(\boldsymbol{w})}{\gamma(\boldsymbol{w})\omega(\eta(\boldsymbol{w}), \theta(\bar{\boldsymbol{y}}, \boldsymbol{z})) + \delta(\boldsymbol{w})}. \tag{8}$$

LaFrance and Pope (2009) present a complete proof of necessity in the case of consumer choice theory. Their proof applies to the current problem with only minor changes

---

[2] An important subset of the literature on this topic includes: Gorman (1953, 1961, 1981); Muellbauer (1975, 1976); Howe, Pollak and Wales (1979); Deaton and Muellbauer (1980); Jorgenson, Lau and Stoker (1980, 1982); Russell (1983, 1996); Jorgenson and Slesnick (1984, 1987); Lewbel (1987, 1988; 1989, 1990, 1991, 2003); Jorgenson (1990); Diewert and Wales (1987, 1988); Blundell (1988);; van Daal and Merkies (1989); Jerison (1993); Russell and Farris (1993, 1998); and Banks, Blundell, and Lewbel (1997), LaFrance, Beatty, Pope and Agnew (2002), LaFrance (2004), LaFrance, Beatty and Pope (2006), and LaFrance and Pope (2009). The focus in the literature has been interior solutions and smooth demand equations. We remain faithful to this approach throughout the present paper.

[3] This result is consistent with exact aggregation as defined by Gorman (1981). One part of our ongoing work is to extend this class to Lau's (1982) definition of exact aggregation, generalizing the left-hand-side of (8) to $f\left(c(\boldsymbol{w}, \bar{\boldsymbol{y}}, \boldsymbol{z})/\pi, \boldsymbol{z}\right)$, wherein cost and quasi-fixed inputs vary across individual economic agents.

in notation. Sufficiency is shown here by considering the structure of the input demands generated by (8). This is accomplished simply enough by differentiating with respect to $\boldsymbol{w}$ and applying Shephard's lemma. To make the notation as compact as possible, let a bold subscript $\boldsymbol{w}$ denote a vector of partial derivatives with respect to the variable input prices and suppress the arguments of the functions $\{\alpha,\beta,\gamma,\delta,\eta,\pi\}$ to yield (after a large amount of straightforward but tedious algebra, which is presented in Appendix B):

$$\begin{aligned}\boldsymbol{x} = \frac{\pi_{\boldsymbol{w}}}{\pi}c + \pi\Big\{&\left[\alpha\beta_{\boldsymbol{w}} - \beta\alpha_{\boldsymbol{w}} + (\alpha^2\lambda+\beta^2)\eta_{\boldsymbol{w}}\right]\frac{1}{f'} \\ &-\left[\alpha\delta_{\boldsymbol{w}} - \delta\alpha_{\boldsymbol{w}} + \gamma\beta_{\boldsymbol{w}} - \beta\gamma_{\boldsymbol{w}} + 2(\alpha\gamma\lambda+\beta\delta)\eta_{\boldsymbol{w}}\right]\frac{f}{f'} \\ &+\left[\gamma\delta_{\boldsymbol{w}} - \delta\gamma_{\boldsymbol{w}} + (\gamma^2\lambda+\delta^2)\eta_{\boldsymbol{w}}\right]\frac{f^2}{f'}\Big\}.\end{aligned} \tag{9}$$

Thus, (8) generates input demands that have the finitely additive and multiplicatively separable structure of any full rank, exactly aggregable system (Gorman 1981; Lau 1982; Lewbel 1989). Note that there are potentially up to four linearly independent variable cost terms on the right with four associated linearly independent vectors of input price functions. Hence, any system generated by (8) will have rank up to, but no greater than four, the highest possible rank (Lewbel 1987, 1990, 1991; LaFrance and Pope 2009).

A third issue when estimating a system of variable input demand equations such as (9) is the fact that quasi-fixed inputs, planned outputs, variable input prices, and total variable cost all are jointly determined with the input demands. Consistent estimation under these conditions is addressed in the empirical application below.

3. The Econometric Cost Model, Data, and Estimates

Previous work at both state and national levels of aggregation with our data set strongly suggests that full rank 3 seriously over-parameterizes the structural model for this data. As a result, we restrict attention here to a rank two model. In this part of the paper, we develop a spatial and temporal econometric model of the conditional demands for 13 variable inputs in U.S. agriculture at the state level: pesticides and herbicides; fertilizer; fuel and natural gas; electricity; purchased feed; purchased seed; purchased livestock; machinery repairs; building repairs; custom machinery services; veterinary services; other materials; and labor. The specification of the variable cost function normalized by the farm wage rate is,

$$\begin{aligned}c_t(\tilde{\boldsymbol{w}}_t, A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t) = &\left[\alpha_{10} + \boldsymbol{\alpha}_1^{\top}\tilde{\boldsymbol{w}}_t\right]A_t + \left[\alpha_{20} + \boldsymbol{\alpha}_2^{\top}\tilde{\boldsymbol{w}}_t\right]K_t \\ &+ \sqrt{\tilde{\boldsymbol{w}}_t^{\top}\boldsymbol{B}\tilde{\boldsymbol{w}}_t + 2\boldsymbol{\gamma}^{\top}\tilde{\boldsymbol{w}}_t + 1} \times \theta(A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t),\end{aligned} \tag{10}$$

where $\boldsymbol{z}_t = [A_t \, K_t \, \boldsymbol{a}_t]^\top$, A_t is farmland, K_t is the value of farm capital, $\boldsymbol{a}_t = [a_{1t} \, a_{2t} \cdots a_{n_y t}]^\top$ is the n_y–vector of acres planted to crops, $A_t = a_{0t} + \boldsymbol{\iota}^\top \boldsymbol{a}_t$, with a_{0t} denoting farmland that is not devoted to crop production, $\bar{\boldsymbol{Y}}_t = [a_{1t}\bar{y}_{1t} \cdots a_{n_y t}\bar{y}_{n_y t}]^\top$ is the n_y–vector of planned crop production, with each element defined as the product of acres planted to the crop times the expected yield per acre, and $\tilde{\boldsymbol{w}}_t = [w_{1t} / w_{n_x t}, \cdots, w_{n_x - 1t} / w_{n_x t}]^\top$ is the $(n_x - 1)$–vector of variable input prices except the farm wage normalized by $w_{n_x t}$.

We treat the n_x^{th} input, labor, asymmetrically with respect to the other inputs both in the structural and stochastic parts of the econometric model. To conserve and simplify notation from this point forward, we drop the ~ over the first n_x–1 input prices, absorb the normalization by w_{n_x} into the notation for variable cost and the n_x–1 first input prices, and define $N = n_x - 1$.

We assume constant returns to scale, so that $\theta(A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t)$ is 1° homogeneous. Define $\alpha_1(\boldsymbol{w}_t) = \alpha_{10} + \boldsymbol{\alpha}_1^\top \boldsymbol{w}_t$, $\alpha_2(\boldsymbol{w}_t) = \alpha_{20} + \boldsymbol{\alpha}_2^\top \boldsymbol{w}_t$, and $\beta(\boldsymbol{w}_t) = \sqrt{\boldsymbol{w}_t^\top \boldsymbol{B} \boldsymbol{w}_t + 2\boldsymbol{\gamma}^\top \boldsymbol{w}_t + 1}$. The necessary and sufficient conditions for the variable cost function to be increasing and concave in the variable input prices throughout an open set containing the data points are as follows (see Appendix C for a complete derivation of the cost function and θ):

Monotonicity in **w**:

$$\begin{aligned}
\frac{\partial c(\boldsymbol{w}_t, A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t)}{\partial \boldsymbol{w}} &= \boldsymbol{\alpha}_1 A_t + \boldsymbol{\alpha}_2 K_t + \frac{\theta}{\beta(\boldsymbol{w}_t)}(\boldsymbol{B}\boldsymbol{w}_t + \boldsymbol{\gamma}) \\
&= \boldsymbol{\alpha}_1 A_t + \boldsymbol{\alpha}_2 K_t + \left[\frac{c_t - \alpha_1(\boldsymbol{w}_t) A_t - \alpha_2(\boldsymbol{w}_t) K_t}{\boldsymbol{w}_t^\top \boldsymbol{B} \boldsymbol{w}_t + 2\boldsymbol{\gamma}^\top \boldsymbol{w}_t + 1} \right] (\boldsymbol{B}\boldsymbol{w}_t + \boldsymbol{\gamma}) \geq \boldsymbol{0};
\end{aligned} \tag{11}$$

Concavity in **w**:

$$\begin{aligned}
\frac{\partial^2 c(\boldsymbol{w}_t, A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t)}{\partial \boldsymbol{w} \partial \boldsymbol{w}^\top} &= \frac{\theta}{\beta(\boldsymbol{w}_t)} \boldsymbol{B} - \frac{\theta}{\beta(\boldsymbol{w}_t)^2} (\boldsymbol{B}\boldsymbol{w} + \boldsymbol{\gamma})(\boldsymbol{B}\boldsymbol{w} + \boldsymbol{\gamma})^\top \\
&= \left[\frac{c(\boldsymbol{w}_t, A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t) - \alpha_1(\boldsymbol{w}_t) A_t - \alpha_2(\boldsymbol{w}_t) K_t}{\boldsymbol{w}_t^\top \boldsymbol{B} \boldsymbol{w}_t + 2\boldsymbol{\gamma}^\top \boldsymbol{w}_t + 1} \right] \times \\
&\quad \left[\boldsymbol{B} - \frac{(\boldsymbol{B}\boldsymbol{w}_t + \boldsymbol{\gamma})(\boldsymbol{B}\boldsymbol{w}_t + \boldsymbol{\gamma})^\top}{(\boldsymbol{w}_t^\top \boldsymbol{B} \boldsymbol{w}_t + 2\boldsymbol{\gamma}^\top \boldsymbol{w}_t + 1)} \right],
\end{aligned} \tag{12}$$

symmetric, negative semi-definite. Setting $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^{\top} + \boldsymbol{\gamma}\boldsymbol{\gamma}^{\top}$, where $\boldsymbol{L}$ is a (lower or upper) triangular matrix with nonzero main diagonal elements implies

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\top} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & \boldsymbol{\gamma} \\ \boldsymbol{0}^{\top} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{L}^{\top} & \boldsymbol{0}^{\top} \\ \boldsymbol{\gamma}^{\top} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}\boldsymbol{L}^{\top} + \boldsymbol{\gamma}\boldsymbol{\gamma}^{\top} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\top} & 1 \end{bmatrix} \tag{13}$$

is positive definite. It follows that $\left[\boldsymbol{B} - \dfrac{(\boldsymbol{B}\boldsymbol{w} + \boldsymbol{\gamma})(\boldsymbol{B}\boldsymbol{w} + \boldsymbol{\gamma})^{\top}}{(\boldsymbol{w}_t^{\top}\boldsymbol{B}\boldsymbol{w}_t + 2\boldsymbol{\gamma}^{\top}\boldsymbol{w}_t + 1)} \right]$ is positive semri-definite and that

$$\begin{bmatrix} \boldsymbol{w}_t^{\top} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\top} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_t \\ 1 \end{bmatrix} = \boldsymbol{w}_t^{\top}\boldsymbol{B}\boldsymbol{w}_t + 2\boldsymbol{\gamma}^{\top}\boldsymbol{w}_t + 1 > 0 \;\forall\; \boldsymbol{w}_t \in \mathbb{R}_+^{n_x - 1}. \tag{14}$$

Given this, the variable cost function is concave in $\boldsymbol{w}$ if and only if

$$c_t(\tilde{\boldsymbol{w}}_t, A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t) < \left[\alpha_0 + \boldsymbol{\alpha}_1^{\top}\tilde{\boldsymbol{w}}_t\right] A_t + \left[(\alpha_2 + \boldsymbol{\alpha}_2^{\top}\tilde{\boldsymbol{w}}_t\right] K_t, \tag{15}$$

(LaFrance, Beatty, and Pope 2006). Hence, we impose $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^{\top} + \boldsymbol{\gamma}\boldsymbol{\gamma}^{\top}$ during estimation and check the monotonicity conditions (11) at all data points once the model is estimated, and find that they are satisfied. We develop the specification for $\theta(A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t)$ in the section on life-cycle consumption and investment decisions and Appendix C.

Applying Shephard's Lemma to (10) and rearranging terms then gives the empirical variable input demand equations in normalized expenditures per dollar of capital as

$$\boldsymbol{e}_t = \boldsymbol{W}_t \left[\boldsymbol{\alpha}_1 \frac{A_t}{K_t} + \boldsymbol{\alpha}_2 + \left(\frac{(c_t/K_t) - \alpha_1(\boldsymbol{w}_t)(A_t/K_t) - \alpha_2(\boldsymbol{w}_t)}{\boldsymbol{w}_t^{\top}\boldsymbol{B}\boldsymbol{w}_t + 2\boldsymbol{\gamma}^{\top}\boldsymbol{w}_t + 1} \right) (\boldsymbol{B}\boldsymbol{w}_t + \boldsymbol{\gamma}) \right] + \boldsymbol{u}_t, \tag{16}$$

where $\boldsymbol{W}_t = \mathbf{diag}[w_{it}]$ is the diagonal matrix with $w_{i,t}$ as the i^{th} main diagonal element and $\boldsymbol{e}_t = [w_{1,t}x_{1,t} \;\cdots\; w_{n_x-1,t}x_{n_x-1,t}]^{\top}$ is the $(n_x - 1)$-vector of normalized expenditures per dollar of capital on all inputs except labor, and we follow standard practice in the empirical analysis of demand systems that apply the Generalized Method of Moments (GMM) and add a vector of random errors to the right-hand-side to obtain the empirical model.

To minimize the potential for aggregation bias with our data. We assume state-specific production technologies for each of the 48 contiguous states in our sample. We also intend to test for embodied, output specific technological change, as well as input specific technological change. Appendix D contains the derivations of these hypotheses.

3.1 General Singular AR(1) Stochastic Processes

We assume that the errors terms for the 12 equations estimated follow to an unrestricted AR(1) process,

$$\boldsymbol{u}_t = \boldsymbol{R}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t,\ \boldsymbol{\varepsilon}_t\ i.i.d.\ (\mathbf{0},\boldsymbol{\Sigma}),\ t = 1,\cdots,T. \tag{17}$$

It is perhaps worthwhile at this point to full explain the way in which w labor asymmetrically in the stochastic part of the model. Let the n random variables, $\boldsymbol{u}_t = [\tilde{\boldsymbol{u}}_t^{\top}\ u_{nt}]^{\top}$, satisfy $u_{nt} = -\boldsymbol{\iota}^{\top}\tilde{\boldsymbol{u}}_t$, where $\tilde{\boldsymbol{u}}_t \in \mathbb{R}^{n-1}$ is an AR(1) stochastic process,

$$\begin{aligned}
&\tilde{\boldsymbol{u}}_t = \boldsymbol{R}\tilde{\boldsymbol{u}}_{t-1} + \tilde{\boldsymbol{\varepsilon}}_t,\\
&E(\tilde{\boldsymbol{\varepsilon}}_t\tilde{\boldsymbol{\varepsilon}}_t^{\top}) = \boldsymbol{\Omega}\ \forall\ t,\\
&E(\tilde{\boldsymbol{u}}_t\tilde{\boldsymbol{\varepsilon}}_{t\pm s}^{\top}) = [\mathbf{0}]\ \forall s,t,\\
&E(\tilde{\boldsymbol{u}}_t\tilde{\boldsymbol{u}}_t^{\top}) = \boldsymbol{\Sigma} = \boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega}\ \forall\ t.
\end{aligned} \tag{18}$$

Then

$$\begin{aligned}
u_{nt} &= -\boldsymbol{\iota}^{\top}\boldsymbol{R}\tilde{\boldsymbol{u}}_{t-1}^{\top} - \boldsymbol{\iota}^{\top}\tilde{\boldsymbol{\varepsilon}}_t,\\
E(u_{nt}^2) &= \boldsymbol{\iota}^{\top}\boldsymbol{\Sigma}\boldsymbol{\iota} = \boldsymbol{\iota}^{\top}(\boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega})\boldsymbol{\iota},\\
E(u_{nt}\tilde{\boldsymbol{u}}_t) &= -\boldsymbol{\Sigma}\boldsymbol{\iota} = -(\boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega})\boldsymbol{\iota},
\end{aligned} \tag{19}$$

and the full $n\times n$ error covariance structure is

$$\begin{aligned}
E\begin{bmatrix} \tilde{\boldsymbol{u}}_t\tilde{\boldsymbol{u}}_t^{\top} & u_{nt}\tilde{\boldsymbol{u}}_t \\ \tilde{\boldsymbol{u}}_t^{\top}u_{nt} & u_{nt}^2 \end{bmatrix} &= \begin{bmatrix} \boldsymbol{\Sigma} & -\boldsymbol{\Sigma}\boldsymbol{\iota} \\ -\boldsymbol{\iota}^{\top}\boldsymbol{\Sigma} & \boldsymbol{\iota}^{\top}\boldsymbol{\Sigma}\boldsymbol{\iota} \end{bmatrix}\\
&= \begin{bmatrix} \boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega} & -(\boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega})\boldsymbol{\iota} \\ -\boldsymbol{\iota}^{\top}(\boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega}) & \boldsymbol{\iota}^{\top}(\boldsymbol{R}\boldsymbol{\Sigma}\boldsymbol{R}^{\top} + \boldsymbol{\Omega})\boldsymbol{\iota} \end{bmatrix}.
\end{aligned} \tag{20}$$

The $n\times n$ AR(1) Markov process satisfies

$$\begin{bmatrix} \tilde{\boldsymbol{u}}_t \\ u_{nt} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \mathbf{0}_{n-1} \\ -\boldsymbol{\iota}^{\top}\boldsymbol{R} & 0 \end{bmatrix}\begin{bmatrix} \tilde{\boldsymbol{u}}_{t-1} \\ u_{nt-1} \end{bmatrix} + \begin{bmatrix} \tilde{\varepsilon}_t \\ \varepsilon_{nt} \end{bmatrix}. \tag{21}$$

The eigen values of this singular stochastic process satisfy

$$\left| \begin{bmatrix} \boldsymbol{R} & \mathbf{0}_{n-1} \\ -\boldsymbol{\iota}^{\mathsf{T}}\boldsymbol{R} & 0 \end{bmatrix} - \lambda \boldsymbol{I}_n \right| = \begin{vmatrix} \boldsymbol{R} - \lambda \boldsymbol{I}_{n-1} & \mathbf{0}_{n-1} \\ -\boldsymbol{\iota}^{\mathsf{T}}\boldsymbol{R} & -\lambda \end{vmatrix} = -\lambda \left| \boldsymbol{R} - \lambda \boldsymbol{I}_{n-1} \right| = 0. \tag{22}$$

Hence, one eigen is 0 and the remaining eigen values are precisely those for $\boldsymbol{R}$. Since labor is the n^{th} input, we model it's time series properties through those for u_{nt} and the adding up condition. This allows the AR(1) matrix of parameters, $\boldsymbol{R}$, to be fully flexible – i.e., there is no requirement that it is symmetric or otherwise restricted. Using national level data, we found this to generate a stationary stochastic process for all variable inputs.

3.2 The Data and Empirical Framework

As noted above, we are applying this model to annual aggregate state-level data on 13 variable inputs in U.S. agriculture (pesticides and herbicides, fertilizer, fuel and natural gas, electricity, purchased feed, purchased seed, purchased livestock, machinery repairs, building repairs, custom machinery services, veterinary services, other materials, and farm labor). The sample period is 1960-2004. This data was compiled by the United States Department of Agriculture's (USDA), Economic Research Service (ERS) and is described in detail in Ball, Halahan, and Nehring (2004). Farmland, equipment, buildings, and structures are treated as quasi-fixed inputs. Hereafter, this data set is called the Ball data.

Due to the way that several variables are constructed in the Ball data, it is necessary to modify and augment this data for empirical implementation. First, we define the replacement cost of owner-operator labor by the farm wage rate. This implies that the return to owner-operator labor in the Ball data due to management skill is treated as a part of the residual claimant's quasi-rent. Second, we use a direct measure of the value of capital obtained from the ERS rather than the measures constructed in the Ball data. Third, estimates of the price of farmland are taken from state-level surveys conducted by the National Agricultural Statistics Service (NASS), rather than the constructed measures in the Ball data. Finally, we adjust the measure of agricultural land. The Census of Agriculture has reported land in farms in four- to five-year intervals for 1954, 1959, 1964, 1969, 1974, 1978, 1982, 1987, 1992, 1997, 2002, and 2002. These are the total farmland numbers used in the sample years that match the Census years. ERS reports the harvested acres for all major crops by state and year since 1947. This data is used to adjust the farmland measures in the Ball data as follows. First, the difference between total farmland in the Ball data and harvested acres is calculated for each non-census year by state. Second, in each period between adjacent censuses, the average of this difference is calculated. This mean difference is treated as fixed in each of the three- or four-year intervals between census years and added to harvested acres to obtain the measure of farmland used in this study in those years of our sample period. We normalize costs, expenditures, and acres by capital rather than total land because we are more confident in the capital measure and Pope, LaFrance and Just (2007) have shown that deflating by a variable that is subject to measurement error leads to difficult econometric issues.

4. Crop Acres, Capital, Savings and Investment, and Consumption in Agriculture

Although the organizational form of farms can vary widely, a recent report by Hoppe and Banker (2006) finds that 98% of U.S. farms remained family farms as of 2003. In a family farm, the entrepreneur controls the means of production and makes investment, consumption, and production decisions. In this section, we develop and analyze a model of the intertemporal nature of these decisions. The starting point is a model similar in spirit to Hansen and Singleton's (1983), but generalized to include consumption decisions and farm investments as well as financial investments and production decisions. The additional variable definitions required for this are as follows:

W_t = beginning-of-period total wealth,

b_t = current holding of bonds with a risk free rate of return r_t,

f_t = current holding of a risky financial asset,

$p_{F,t}$ = beginning-of-period market price of the financial asset,

$\rho_{F,t+1}$ = dividend plus capital gains rate on the financial asset,

$a_{i,t}$ = current allocation of land to the i^{th} crop, $i = 1,\ldots,n_Y$,

A_t = total quantity of farm land,

$p_{L,t}$ = beginning-of-period market price of land,

$\rho_{L,t+1} = (p_{L,t+1} - p_{L,t}) / p_{L,t}$ = capital gain rate on land,

$\overline{y}_{i,t}$ = expected yield per acre for the i^{th} crop, $i = 1,\ldots,n_Y$,

$y_{i,t+1}$ = realized yield of the i^{th} crop,

$p_{Y_i,t+1}$ = end-of-period realized market price for the i^{th} farm product,

$\boldsymbol{q}_t$ = vector of quantities of consumption goods,

$\boldsymbol{p}_{Q,t}$ = vector of market prices for consumer goods,

m_t = total consumption expenditures,

$u(\boldsymbol{q}_t)$ = periodic utility from consumption.

As with all discrete time models, timing can be represented in multiple ways. In the model used here, all financial returns and farm asset gains are assumed to be realized at the end of each time period (where depreciation is represented by a negative asset gain). Variable inputs are assumed to be committed to farm production activities at the beginning of each decision period and the current period market prices for the variable inputs are known when these use decisions are made. Agricultural production per acre is realized stochastically at the end of the period such that

$$y_{i,t+1} = \bar{y}_{i,t}(1+\varepsilon_{i,t+1}),\ i = 1,\ldots,n_Y\ , \tag{23}$$

where $\varepsilon_{i,t+1}$ is a random output shock with $E(\varepsilon_{i,t+1}) = 0$. Consumption decisions are made at the beginning of the decision period and the current market prices of consumption good are known when these purchases are made. Utility is assumed to be strictly increasing and concave in $\boldsymbol{q}_t$. The total beginning-of-period quantity of land is $A_t = \boldsymbol{\iota}^{\top}\boldsymbol{a}_t$, with $\boldsymbol{\iota}$ denoting an n_Y–vector of ones. Homogeneous land is assumed with a scalar price, $p_{L,t}$.

To simplify our derivations, we require an uncommon piece of matrix notation. The Hadamard/Schur product of two $n\times m$ matrices **A** and **B** is the matrix whose elements are element-by-element products of the elements of **A** and $\boldsymbol{B}$, $\boldsymbol{A}\bullet\boldsymbol{B} = \boldsymbol{C} \Leftrightarrow c_{ij} = a_{ij}b_{ij}\ \forall i, j$. This definition assists the derivation of the arbitrage conditions present in what follows.

Revenue at $t + 1$ is the random price times production

$$R_{t+1} = \sum_{i=1}^{n_Y}(p_{Y_i,t+1}\bar{y}_{i,t}a_{i,t}(1+\varepsilon_{i,t+1})) \equiv (\boldsymbol{p}_{Y,t+1}\bullet\boldsymbol{a}_t\bullet\bar{\boldsymbol{y}}_t)^{\top}(\boldsymbol{\iota}+\boldsymbol{\varepsilon}_{t+1}). \tag{24}$$

Wealth is allocated at the beginning of period t to investments, the variable cost of production, and consumption,

$$W_t = b_t + f_t + p_{L,t}A_t + K_t + c_t(\boldsymbol{w}_t, \boldsymbol{a}_t, K_t, \bar{\boldsymbol{Y}}_t) + m_t. \tag{25}$$

Although some costs occur at or near harvest (near $t + 1$), we include all costs in (25) at time t because they are incurred before revenues are received. Consumer utility maximization yields the indirect utility function conditioned on consumer good prices and consumption expenditure,

$$\upsilon(\boldsymbol{p}_{Q,t}, m_t) \equiv \max_{\boldsymbol{q}\in R_+^{nQ}}\left\{u(\boldsymbol{q}) : \boldsymbol{p}_{Q,t}^{\top}\boldsymbol{q} = m_t\right\}. \tag{26}$$

Realized end of period wealth is

$$\begin{aligned} W_{t+1} &= (1+r_t)b_t + (1+\rho_{F,t+1})f_t + (1+\rho_{L,t+1})p_{L,t}A_t \\ &\quad + (1+\rho_{K,t+1})K_t + (\boldsymbol{p}_{Y,t+1}\bullet\boldsymbol{a}_t\bullet\bar{\boldsymbol{y}}_t)^{\top}(\boldsymbol{\iota}+\boldsymbol{\varepsilon}_{t+1}), \end{aligned} \tag{27}$$

where $\rho_{K,t+1}$ is the proportional change in the value of capital held at the beginning of the production period. Thus, the decision maker's wealth is increased by net returns on assets and farm revenue. The owner/operator decision maker's intertemporal utility function is assumed to be

$$U_T(\boldsymbol{q}_1,...,\boldsymbol{q}_T) = \sum_{t=0}^{T}(1+\rho)^{-t}u(\boldsymbol{q}_t)\,. \tag{28}$$

The producer is assumed to maximize von Neumann-Morgenstern expected utility of the discounted present value of the periodic utility flows from goods consumption.

By Euler's theorem, constant returns to scale implies linear homogeneity of the variable cost function in capital, land, and output. For the variable cost function derived and estimated in this paper, this implies

$$\begin{aligned} c_t(\boldsymbol{w}_t,\boldsymbol{a}_t,A_t,K_t,\overline{\boldsymbol{Y}}_t) \equiv{}& \frac{\partial c_t(\boldsymbol{w}_t,\boldsymbol{a}_t,A_t,K_t,\overline{\boldsymbol{Y}}_t)}{\partial \boldsymbol{a}_t^{\top}}\boldsymbol{a}_t + \frac{\partial c_t(\boldsymbol{w}_t,\boldsymbol{a}_t,A_t,K_t,\overline{\boldsymbol{Y}}_t)}{\partial A_t}A_t \\ &+ \frac{\partial c_t(\boldsymbol{w}_t,\boldsymbol{a}_t,A_t,K_t,\overline{\boldsymbol{Y}}_t)}{\partial K_t}K_t + \frac{\partial c_t(\boldsymbol{w}_t,\boldsymbol{a}_t,A_t,K_t,\overline{\boldsymbol{Y}}_t)}{\partial \overline{\boldsymbol{Y}}_t^{\top}}\overline{\boldsymbol{Y}}_t. \end{aligned} \tag{29}$$

The vector of expected crop outputs satisfies

$$\overline{\boldsymbol{Y}}_t = \overline{\boldsymbol{y}}_t \bullet \boldsymbol{a}_t, \tag{30}$$

where $\overline{y}_{j,t}$ is the expected yield per acre and $a_{j,t}$ is the number of acres planted for the j^{th} crop. The variable cost function might depend on time due to technological change or other dynamic forces, and the subscript t indicates this possibility. To distinguish quasi-fixed from variable inputs and to account for the possibility of hysteresis in agricultural investments, we allow for adjustment costs for total farmland and capital,

$$C_{Adj}(A_t - A_{t-1}, K_t - K_{t-1}) = ½\gamma_A(A_t - A_{t-1})^2 + ½\gamma_K(K_t - K_{t-1})^2, \tag{31}$$

with $\gamma_A, \gamma_K \geq 0$.

This problem is solved by stochastic dynamic programming working backwards recursively from the last period in the planning horizon to the first. In the last period, the optimal decision is to invest or produce nothing and consume all remaining wealth, i.e., $m_T = W_T$. Denote the last period's optimal value function by $v_T(W_T, A_{T-1}, K_{T-1})$. Then $v_T(W_T, A_{T-1}, K_{T-1}) = \upsilon(\boldsymbol{p}_{Q,T}, W_T)$ is the optimal utility for the terminal period. For all other time periods, stochastic dynamic programming yields the Bellman backward recursion (Bellman and Dreyfus 1962). For an arbitrary $t < T$, the Lagrangean for the problem at time *t* is

$$\begin{aligned}\ell_t &= \upsilon(\boldsymbol{p}_{Q,t}, m_t) + (1+r)^{-1} E_t \Big\{ V_{t+1}\Big[(1+r)b_t + (1+\rho_{F,t+1}) f_t \\ &+ p_{L,t+1}A_t + (1+\rho_{K,t+1})K_t + (\boldsymbol{p}_{Y,t+1}\bullet \overline{\boldsymbol{y}}_t \bullet \boldsymbol{a}_t)^{\top}(\boldsymbol{\iota} + \boldsymbol{\varepsilon}_{t+1}), A_t, K_t \Big]\Big\} \\ &+ \lambda_t \Big\{ W_t - m_t - b_t - f_t - p_{L,t}A_t - K_t - c_t(\boldsymbol{w}_t, \boldsymbol{a}_t, A_t, K_t, \overline{\boldsymbol{y}}_t \bullet \boldsymbol{a}_t) \\ &- \tfrac{1}{2}\gamma_A (A_t - A_{t-1})^2 - \tfrac{1}{2}\gamma_K (K_t - K_{t-1})^2 \Big\} + \mu_t (A_t - \boldsymbol{\iota}^{\top}\boldsymbol{a}_t),\end{aligned} \tag{32}$$

where $E_t(\bullet)$ is the conditional expectation at the beginning of period *t* given information available at that point in time, λ_t is the shadow price for the beginning-of-period wealth allocation constraint, and μ_t is the shadow price for the land allocation constraint. The first-order, necessary and sufficient Kuhn-Tucker conditions are the two constraints and the following:

$$\frac{\partial \ell_t}{\partial m_t} = \frac{\partial \upsilon_t}{\partial m_t} - \lambda_t \le 0,\ m_t \ge 0,\ m_t \frac{\partial \ell_t}{\partial m_t} = 0; \tag{33}$$

$$\frac{\partial \ell_t}{\partial b_t} = E_t\left(\frac{\partial V_{t+1}}{\partial W_{t+1}} \right) - \lambda_t \le 0,\ b_t \ge,\ b_t \frac{\partial \ell_t}{\partial b_t} = 0; \tag{34}$$

$$\frac{\partial \ell_t}{\partial f_t} = (1+r)^{-1} E_t\left[\frac{\partial V_{t+1}}{\partial W_{t+1}} (1+\rho_{F,t+1}) \right] - \lambda_t \le 0,\ f_t \ge 0,\ f_t \frac{\partial \ell_t}{\partial f_t} = 0. \tag{35}$$

$$\begin{aligned}\frac{\partial \ell_t}{\partial A_t} &= (1+r)^{-1} E_t\left(\frac{\partial V_{t+1}}{\partial W_{t+1}} p_{L,t+1} + \frac{\partial V_{t+1}}{\partial A_t} \right) \\ &-\lambda_t \left[p_{L,t} + \frac{\partial c_t}{\partial A_t} + \gamma_A (A_t - A_{t-1}) \right] + \mu_t \le 0,\ A_t \ge 0,\ A_t \frac{\partial \ell_t}{\partial A_t} = 0;\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial \ell_t}{\partial K_t} &= (1+r)^{-1} E_t\left[\frac{\partial V_{t+1}}{\partial W_{t+1}} (1+\rho_{K,t+1}) + \frac{\partial V_{t+1}}{\partial K_{t+1}} \right] \\ &- \lambda_t \left[1 + \frac{\partial c_t}{\partial K_t} + \gamma_K (K_t - K_{t-1}) \right] \le 0, K_t \ge 0, K_t \frac{\partial \ell_t}{\partial K_t} = 0;\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial \ell_t}{\partial \boldsymbol{a}_t} &= (1+r)^{-1} E_t\left[\frac{\partial V_{t+1}}{\partial W_{t+1}} (\boldsymbol{p}_{Y,t+1}\bullet \overline{\boldsymbol{y}}_t)\bullet(\boldsymbol{\iota} + \boldsymbol{\varepsilon}_{t+1}) \right] - \lambda_t \left(\frac{\partial c_t}{\partial \boldsymbol{a}_t} + \frac{\partial c_t}{\partial \overline{\boldsymbol{Y}}_t} \bullet \overline{\boldsymbol{y}}_t \right) - \mu_t \boldsymbol{\iota} \le 0, \\ &\boldsymbol{a}_t \ge \boldsymbol{0},\ \boldsymbol{a}_t^{\top} \frac{\partial \ell_t}{\partial \boldsymbol{a}_t} = 0;\end{aligned} \tag{38}$$

$$\frac{\partial \ell_t}{\partial \overline{\boldsymbol{y}}_t} = (1+r)^{-1} E_t \left[\frac{\partial V_{t+1}}{\partial W} \boldsymbol{p}_{Y,t+1} \bullet \boldsymbol{a}_t \bullet (\boldsymbol{\imath} + \boldsymbol{\varepsilon}_{t+1}) \right] - \lambda_t \frac{\partial c_t}{\partial \overline{\boldsymbol{Y}}_t} \bullet \boldsymbol{a}_t \leq \mathbf{0},$$

$$\overline{\boldsymbol{y}}_t \geq \mathbf{0},\ \overline{\boldsymbol{y}}_t^{\top} \frac{\partial \ell_t}{\partial \overline{\boldsymbol{y}}_t} = 0. \tag{39}$$

We also have the following implications of the envelope theorem:

$$\begin{aligned} \frac{\partial V_t}{\partial W_t} &= \lambda_t; \\ \frac{\partial V_t}{\partial A_{t-1}} &= \lambda_t \gamma_A (A_t - A_{t-1}); \\ \frac{\partial V_t}{\partial K_{t-1}} &= \lambda_t \gamma_K (K_t - K_{t-1}); \end{aligned} \tag{40}$$

where the variables $\{\lambda_t, A_t, K_t\}$ are all evaluated at their optimal choices.

Combining the Kuhn-Tucker conditions with the results of the envelope theorem and assuming an interior solution for consumption, bonds, and risky financial assets, we obtain the standard Euler equations for smoothing the marginal utility of consumption and wealth,

$$\frac{\partial \upsilon_t}{\partial m_t} = E_t \left(\frac{\partial \upsilon_{t+1}}{\partial m_{t+1}} \right) = \frac{\partial V_t}{\partial W_t} = E_t \left(\frac{\partial V_{t+1}}{\partial W_{t+1}} \right) = \lambda_t = E_t(\lambda_{t+1}), \tag{41}$$

and the standard arbitrage condition for excess returns to risky financial assets,

$$E_t \left[(\rho_{F,t+1} - r) \frac{\partial V_{t+1}}{\partial W_{t+1}} \right] = 0 \tag{42}$$

The complementary slackness of the Kuhn-Tucker condition (39), implies that for each crop we have the supply condition under risk,

$$E_t \left[\frac{\partial V_{t+1}}{\partial W_{t+1}} \left(p_{Y_i,t+1} - (1+r) \frac{\partial c_t}{\partial \overline{Y}_{i,t}} \right) \right] \overline{Y}_{i,t} = 0,\ i = 1, \cdots, n_y. \tag{43}$$

For each crop produced in positive quantity, this reduces to the well-known result that the conditional covariance between the marginal utility of future wealth and the difference between the ex post realized market price the marginal cost of production must vanish. The multiplicative factor $1+r$ is multiplied by ex ante marginal cost so that these two economic values are measured at a common point in time – in the present case at the end of the production period.

To obtain the arbitrage condition for the level of investment in agriculture, we combine the linear homogeneity property of the variable cost function in $(\boldsymbol{a}_t, A_t, K_t, \bar{\boldsymbol{Y}}_t)$ from equation (29) with complementary slackness in Kuhn-Tucker conditions (37)–(41),

$$0 = \frac{\partial \ell_t}{\partial \boldsymbol{a}_t^{\top}} \boldsymbol{a}_t + \frac{\partial \ell_t}{\partial A_t} A_t + \frac{\partial \ell_t}{\partial K_t} K_t, \tag{44}$$

which, after considerable rearranging and combining of terms, gives

$$\begin{aligned} E_t \{ \partial V_{t+1} / \partial W_{t+1} [s_{K,t}(\rho_{K,t+1} - r) + s_{L,t}(\rho_{L,t+1} - r) + \pi_{t+1} \\ + s_{K,t}\gamma_K (K_{t+1} - (2+r)K_t + (1+r)K_{t-1}) \\ + s_{A,t}\gamma_A (A_{t+1} - (2+r)A_t + (1+r)A_{t-1})] \} = 0, \end{aligned} \tag{45}$$

where $s_{K,t} = K_t / (p_{L,t}A_t + K_t)$ is capital's share of the value of the investment in agriculture in period t, $s_{L,t} = p_{L,t}A_t / (p_{L,t}A_t + K_t)$ is land's share of the value of the investment in agriculture in period t, $s_{A,t} = A_t / (p_{L,t}A_t + K_t)$ is the ratio of the quantity of land to the value of the investment in agriculture at the beginning of the production period, and

$$\pi_{t+1} = \frac{R_{t+1} - (1+r)c_t}{p_{L,t}A_t + K_t} \tag{46}$$

is the *ex post* net return to crop production over the variable cost of production relative to the *ex ante* value of agricultural investment, so that it is measured as a rate of return to agricultural production. The first 3 terms inside of the square brackets of equation (45) represent the total sum of the excess returns to agriculture, including the rate of net return to crop production over variable costs. The last two terms in square brackets capture the effects of adjustment costs for farm capital and farmland. This has the standard one-period ahead and one period behind 2^{nd}–order difference structure common to quadratic adjustment cost models in dynamic optimization problems.

To implement this system of Euler equations, we assume that the indirect utility function for consumption goods is a member of the certainty equivalent class,

$$\upsilon(\boldsymbol{p}_{Qt}, m_t) = \frac{m_t}{\pi_C(\boldsymbol{p}_{Qt})} - ½\beta \left(\frac{m_t}{\pi_C(\boldsymbol{p}_{Qt})} \right)^2, \tag{47}$$

where $0 \le \beta < \pi_C(\boldsymbol{p}_{Qt}) / m_t \ \forall\ t$ and $\pi_C(\boldsymbol{p}_{Qt})$ is the consumer price index (CPI) for all items. Then the marginal utility of money in each period is

$$\lambda_t = \frac{1-\beta\left[m_t/\pi_C(\boldsymbol{p}_{Qt})\right]}{\pi_C(\boldsymbol{p}_{Qt})}. \tag{48}$$

This allows us to identify the effects of risk aversion separately from those of adjustment costs and hysteresis in agricultural investment decisions. We assume that the preferences of agricultural producers are of the same class as all other individuals in the economy. This allows use of the observable variable per capita personal consumption expenditure, rather than the latent variable wealth, to model the empirical arbitrage equations.

Empirical Arbitrage Equations and Data

Let $n \le n_y$ be the number of crops included in the empirical model. The specification that we choose for $\partial c_t / \partial \bar{Y}_{i,t}$ is (see Appendix C for a complete derivation),

$$\begin{gathered}\frac{\partial c_t}{\partial \bar{Y}_{i,t}} = \beta(\boldsymbol{w}_t)\left(\theta_i + \sum_{j=1}^{n_x}\theta_{ij}\frac{\bar{Y}_{j,t}}{K_t}\right), \text{ with} \\ \beta(\boldsymbol{w}_t) = \sqrt{\tilde{\boldsymbol{w}}_t^{\top}\hat{\boldsymbol{B}}\tilde{\boldsymbol{w}}_t + 2\hat{\boldsymbol{\gamma}}^{\top}\tilde{\boldsymbol{w}}_t w_{n_x,t} + w_{n_x,t}^2}.\end{gathered} \tag{49}$$

The $n+3$ empirical arbitrage/Euler equations therefore are

$$\begin{aligned}
&\textbf{Consumption / Bonds}: && \beta(m_{t+1}-m_t) = u_{1,t+1}, \\
&\textbf{Risky Assets}: && (1-\beta m_{t+1})(\rho_{F,t+1}-r) = u_{2,t+1}, \\
&\textbf{Crops}: && (1-\beta m_{t+1})\left[p_{Y_i,t+1} - (1+r)\widehat{\beta(\boldsymbol{w}_t)}\left(\theta_i + \sum_{j=1}^{n}\frac{\theta_{ij}Y_{j,t+1}}{K_t}\right)\right] = u_{i,t+1}, \\
& && i = 3,\cdots,n+2, \qquad\qquad (50)\\
&\textbf{Agriculture}: && (1-\beta m_{t+1})\left[s_{K,t}(\rho_{K,t+1}-r) + s_{L,t}(\rho_{L,t+1}-r) + \pi_{t+1}\right. \\
& && + s_{A,t}\gamma_A\left(A_{t+1}-(2+r)A_t+(1+r)A_{t-1}\right) \\
& && \left. + s_{K,t}\gamma_K\left(K_{t+1}-(2+r)K_t+(1+r)K_{t-1}\right)\right] = u_{n+3,t+1}.
\end{aligned}$$

The instruments we will use are variable cost per unit of capital, land per unit of capital, and variable input prices all lagged two periods, plus the following general economy variables lagged one period: real per capita disposable personal income; unemployment rate; the real rate of return on AAA corporate 30-year bonds; real manufacturing wage rate; real index of prices paid by manufacturers for materials and components; and real index of prices paid by manufacturers for fuel, energy and power. Per capita disposable personal income is deflated by the consumer price index for all items. The aggregate wholesale price variables are deflated by the implicit price deflator for gross domestic product. The real rate of return on corporate bonds is calculated as the nominal rate of return minus the midyear annual inflation rate.

The single equation and system-wide 1st - and 2nd-order Brownian bridge tests for specification error and parameter instability developed in LaFrance (2008) will be used to check the model for robustness to misspecification errors and non-constant parameters. Appendix E presents and discusses this set of within-sample residual test statistics.

5. Econometric Structure

Let $i = 1,\cdots,I$ index states, $j = 1,\cdots,N$ index equations, and $t = 1,\cdots,T$ index time. In general, the state-level equations can be written as

$$x_{ijt} = f_{ij}(\boldsymbol{w}_{it}, k_{it}, c_{it}, t;\boldsymbol{\theta}) + u_{ijt},\ i = 1,\cdots,I,\ j = 1,\cdots,N,\ t = 1,\cdots,T, \tag{51}$$

where $\boldsymbol{w}_{it}$ is the $N\times 1$ vector of (normalized) input prices, k_{it} is capital per acre, c_{it} is (normalized) variable cost per acre, $\boldsymbol{\theta}$ is a $K\times 1$ vector of parameters to be estimated, and u_{ijt} is a mean zero random error term. These errors have intertemporal autocorrelation,

$$u_{ijt} = \sum_{j'=1}^{N} \phi_{jj'} u_{ij't-1} + v_{ijt},\ i = 1,\cdots,I,\ j = 1,\cdots,N,\ t = 1,\cdots,T. \tag{52}$$

The mean zero random variables v_{ijt} are uncorrelated across time and correlated across inputs, $E\left(\boldsymbol{v}_{i\bullet t}\boldsymbol{v}_{i\bullet t}^{\mathsf{T}}\right) = \boldsymbol{\Sigma}$, $\boldsymbol{v}_{i\bullet t} = [v_{i1t}\cdots v_{iNt}]^{\mathsf{T}}$. Let $\boldsymbol{\Sigma}^{-1} = \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}$, so that the typical element of $\boldsymbol{\varepsilon}_{i\bullet t} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{v}_{i\bullet t} = \boldsymbol{L}^{\mathsf{T}}\boldsymbol{v}_{i\bullet t}$ can be written as $\varepsilon_{ijt} = \sum_{j'=1}^{N} \ell_{jj'} v_{ij't}$. These mean zero, unit variance random variables, ε_{ijt}, are uncorrelated across both inputs and time, but are correlated across space, $E(\varepsilon_{ijt}\varepsilon_{i'jt}) = \rho(d_{ii'}),\ j = 1,\cdots,N,$, where $d_{ii'}$ is the geographic distance between states i and i', $\rho(0) = 1$ and the $I\times I$ matrix,

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho(d_{12}) & \cdots & \rho(d_{1I}) \\ \rho(d_{12}) & 1 & \cdots & \rho(d_{2I}) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(d_{1I}) & \rho(d_{2I}) & \cdots & 1 \end{bmatrix}, \tag{53}$$

is symmetric, positive definite. For simplicity, we assume that $\boldsymbol{R}$ is constant across j.

5.1 Consistent, Efficient Estimation

Let $\boldsymbol{Z}_i$ denote the matrix of instruments for each state and let $\mathbb{N}_i = \boldsymbol{Z}_i(\boldsymbol{Z}_i^{\mathsf{T}}\boldsymbol{Z}_i)^{-1}\boldsymbol{Z}_i^{\mathsf{T}}$ the projection matrix for the instruments. Let $\boldsymbol{\tau} = [12\cdots T]^{\mathsf{T}}$, stack equation (51) by inputs and time, and use NL2SLS to estimate $\boldsymbol{\theta}$ consistently,

$$\hat{\boldsymbol{\theta}}_{2SLS} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \sum_{i=1}^{I} \left[\boldsymbol{x}_{i\bullet\bullet} - \boldsymbol{f}_{i\bullet}(\boldsymbol{w}_{i\bullet\bullet}, \boldsymbol{k}_{i\bullet}, \boldsymbol{c}_{i\bullet}, \boldsymbol{\tau};\boldsymbol{\theta})\right]^{\mathsf{T}} \left(\mathbb{N}_i \otimes \boldsymbol{I}_N\right)\left[\boldsymbol{x}_{i\bullet\bullet} - \boldsymbol{f}_{i\bullet}(\boldsymbol{w}_{i\bullet\bullet}, \boldsymbol{k}_{i\bullet}, \boldsymbol{c}_{i\bullet}, \boldsymbol{\tau};\boldsymbol{\theta})\right]. \quad (54)$$

Use this consistent estimator of $\boldsymbol{\theta}$ to generate consistent estimates of the errors,

$$\hat{u}_{ijt} = x_{ijt} - f_{ij}(\boldsymbol{w}_{it}, k_{it}, c_{it}, t; \hat{\boldsymbol{\theta}}_{2SLS}),\ i = 1,\cdots,I,\ \ j = 1,\cdots,N,\ t = 1,\cdots,T. \quad (55)$$

For $t = 2,\cdots,T$, estimate the 3×3 intertemporal correlation matrix, $\boldsymbol{\Phi}$, consistently by linear SUR,

$$\hat{\boldsymbol{\Phi}} = \underset{\boldsymbol{\Phi}}{\arg\min} \left\{ \sum_{i=1}^{N}\sum_{t=2}^{T} (\hat{\boldsymbol{u}}_{i\bullet t} - \boldsymbol{\Phi}\hat{\boldsymbol{u}}_{i\bullet t-1})^{\mathsf{T}} \hat{\boldsymbol{\Sigma}}^{-1} (\hat{\boldsymbol{u}}_{i\bullet t} - \boldsymbol{\Phi}\hat{\boldsymbol{u}}_{i\bullet t-1}) \right\}. \quad (56)$$

We could start this stage with $\hat{\boldsymbol{\Sigma}} = \boldsymbol{I}$ and iterate once on the covariance matrix, or with $\hat{\boldsymbol{\Sigma}}$ calculated from the 2SLS estimates for $\boldsymbol{\theta}$ and $\boldsymbol{\Phi} = [\mathbf{0}]$. Either approach gives a consistent estimator for $\boldsymbol{\Phi}$, since the weight matrix does not affect consistency. But the former method is the preferred method if the above model assumptions are true.

Construct consistent estimates of the spatially correlated error terms,

$$\hat{\varepsilon}_{ijt} = \sum\nolimits_{j'=1}^{N} \hat{\ell}_{jj'} \hat{v}_{ij't}\,, \quad (57)$$

where $\hat{v}_{ijt} = \hat{u}_{ijt} - \sum\nolimits_{j'=1}^{N} \hat{\phi}_{jj'} \hat{u}_{ij't-1}$ and $\hat{\boldsymbol{L}} = [\hat{\ell}_{jj'}]$ satisfies $\tilde{\boldsymbol{\Sigma}}^{-1} = \hat{\boldsymbol{L}}\hat{\boldsymbol{L}}^{\mathsf{T}}$.

Calculate the consistent sample estimates for the spatial correlations as,

$$\hat{\rho}_{ii'} = \sum\nolimits_{j=1}^{N} \sum\nolimits_{t=2}^{T} \hat{\varepsilon}_{ijt}\hat{\varepsilon}_{i'jt} \big/ N(T-1),\ \ i, i' = 1,\cdots,I. \quad (58)$$

Estimate the relationship for the sample estimates and the geographic distance between states using robust least squares to obtain $\hat{\boldsymbol{R}} = \left[\hat{\rho}(d_{ii'})\right]$.

Let $\hat{\boldsymbol{R}}^{-1} = \boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}$ and write $\omega_{ijt} \approx \sum\nolimits_{i'=1}^{I} \hat{q}_{ii'}\varepsilon_{i't}$. Now (in theory, or principle), the random variables ω_{ijt} are mean zero, unit variance, and uncorrelated across inputs, states, and time. Substitute back recursively, so that

$$\begin{aligned}\omega_{ijt} &\approx \sum_{i'=1}^{I} \hat{q}_{ii'} \varepsilon_{i'jt} \\ &\approx \sum_{i'=1}^{I} \hat{q}_{ii'} \sum_{j'=1}^{N} \hat{\ell}_{jj'} \nu_{i'j't} \\ &\approx \sum_{i'=1}^{I} \hat{q}_{ii'} \sum_{j'=1}^{N} \hat{\ell}_{jj'} \left(u_{i'j't} - \sum\nolimits_{j''=1}^{N} \hat{\phi}_{j'j''} u_{i'j''t-1} \right).\end{aligned} \tag{59}$$

Equation (51) gives the definition required to write the last line of equation (59) in terms of observable variables, consistently estimated covariance parameters, and the structural parameters, say $\omega_{ijt}(\boldsymbol{\theta})$. A final NL3SLS step of the form,

$$\hat{\boldsymbol{\theta}}_{3SLS} = \arg\min_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{I} \boldsymbol{\omega}_{i\bullet\bullet}(\boldsymbol{\theta})^{\top} \left(\mathbb{N}_i \otimes \boldsymbol{I}_N \right) \boldsymbol{\omega}_{i\bullet\bullet}(\boldsymbol{\theta}) \right\}, \tag{60}$$

gives consistent, efficient, asymptotically normal estimates of $\boldsymbol{\theta}$. White's heteroskedasticity consistent covariance estimator can be used for robustness to any remaining sources of herteroskedasticity beyond the state-specific variance-covariance matrices.

6. Conclusions

This paper has developed a new structural model of variable input use, production, acreage allocations, capital investment, and consumption choices in the U.S. farm sector. The theoretical framework identifies and incorporates the restrictions that are necessary and sufficient to estimate variable input use using only observable data, and to aggregate from micro units of behavior to county-, state-, region-, or country-levels of data and analyses. We defined, specified and estimated a dynamic life-cycle model of decision making under risk. We disciplined the model and associated parameter estimates for risk aversion in agricultural production and investment decisions with the interactions that naturally occur among the available alternative investment and savings opportunities in the economy.

Current work applies this to state-level data, which should to mitigate the issues related to aggregating across different production regions, climates, and output choice sets. We incorporate input and output specific technological change in the empirical model, which should help address issues due to specification errors and structural change that cannot be captured in the aggregate setup. We are specifying and estimating the variable input use decisions and the asset management choices simultaneously to exploit cross-equation parameter restrictions and increase the efficiency of our parameter estimates. And last, the data set is in the final stages of being updated to the 21st century, which will make the model and empirical analysis more timely and relevant to current farm policies.

One of the central issues guiding agricultural policy is how risk affects choice and welfare. Here, that is manifest in the movement towards general equilibrium found in the cross-moment equations in (46) and the cost structure in (45). This provides a rich mechanism for policy analysis. The conventional agricultural focus is how policies affect the risk environment and thereby production choice and welfare. Thus, for example, in a partial equilibrium model of the farm sector, one often studies the effects of a particular policy on the risk environment on the portfolio of crop choice (Chavas and Holt 1996). Here, it is clear that the *evolution* of wealth and income in all forms, and consumption, "cause" production choices. Although this point is not new (e.g., Wright and Hewitt 1994), it has not been formally modeled and estimated.

With the results of this analys, one can trace the effects of any policy altering the distribution of agricultural crop income on the choices which restores equilibrium. More specifically, it means that significant responses may be outside of agriculture by changing non-agricultural investment and consumption. These responses likely will alter the normative and positive conclusions of the effects of policies substantially.

Indeed, returning to the example of crop insurance discussed in the introduction, the social value of public insurance will likely be reduced as more margins for adjustment (arbitrage conditions) are included in the analysis. In contrast, an increase in uncertainty (the covariance term) in non-agricultural investments as witnessed recently could increase the demand for risk reducing agricultural instruments. The key point is that unless one has a model that provides for these interactions, one will not obtain reasonable policy conclusions.

The second general policy insight that can be obtained here is a distinction between long-run and shorter-run effects which has been one of the foundations of agricultural policy analyses, preceding the seminal work of Nerlove. Yet models in current vogue can only be interpreted as long-run analyses where adjustment costs are zero. This means that one has a natural structural way in the current model to distinguish short-run and long-run elasticities. For example, this implies that policies that raise the return to insurance (e.g., through public subsidies) have larger responses in the long-run than in the short-run.

References

Akridge, J.T. and T. W. Hertel. 1986. "Multiproduct Cost Relationships for Retail Fertilizer Plants." *American Journal of Agricultural Economics* 68: 928-38.

Babcock, B.A. and D. Hennessy. 1996. "Input Demand under Yield and Revenue Insurance." *American Journal of Agricultural Economics.* 78: 416-27.

Ball, E., Cavazos, R., LaFrance, J.T., Pope, R., and Tack. J. 2010. "Aggregation Theory and Arbitrage in Joint Production." In E. Ball, ed., *The Economic Impact of Public Support to Agriculture.* New York: Elsevier-Science, in press.

Ball, V.E., C. Hallahan, and R. Nehring. 2004. "Convergence of Productivity: An Analysis of the Catch-up Hypothesis within a Panel of States." *American Journal of Agricultural Economics* 86: 1315-3121.

Bellman, R., and S. Dreyfus. 1962. *Applied Dynamic Programming,* Princeton: Princeton University Press.

Blackorby, C., Primont, D. and R. Russell. 1977. "Dual Price and Quantity Aggregation." *Journal of Economic Theory* 14: 130-48.

_____. 1978. *Separability and Functional Structure: Theory and Economic Applications.* New York: American Elsevier/North-Holland.

Chambers, R.G. 1984. "A Note on Separability of the Indirect Production Function and Measures of Substitution." *Southern Economic Journal* 4: 1189-91.

_____. 1988. *Applied Production Analysis.* Cambridge: Cambridge University press.

_____. 1989. "Insurability and Moral Hazard in Agricultural Insurance Markets." *American Journal of Agricultural Economics.* 71: 604-616.

Chambers, R. G. and R. D. Pope. 1991. "Testing for Consistent Aggregation." *American Journal of Agricultural Economics*." 73: 808-18.

_____. 1994. "A Virtually Ideal Production System: Specifying and Estimating the VIPS Model" *American Journal of Agricultural Economics* 76: 105-13.

Chambers, R.G. and J. Quiggin. 2000. *Uncertainty, Production, choice and Agency: The State-Contingent Approach.* Cambridge: Cambridge University Press.

Chavas, J.P. 2008. "A Cost Approach to Economic Analysis under State-Contingent Production Uncertainty." *American Journal of Agricultural Economics* 90: 435-466.

Chavas J.-P. and Holt., M. 1996. "Economic Behavior under Uncertainty: a Joint Analysis of Risk and Technology". *Review of Economics and Statistics* 51, 329-335.

Christensen, L.R., D.W. Jorgenson, and L.J. Lau. 1975. "Transcendental Logarithmic Utility Functions." *American Economic Review* 65: 367-83.

Deaton, A. and Muellbauer, J. 1980. "An Almost Ideal Demand system." *American Economic Review*, 70: 312-26.

Diewert, W.E. and T.J. Wales. 1987. "Flexible Functional Forms and Global Curvature Conditions." *Econometrica* 55: 43-68.

_____. 1988. "Normalized Quadratic Systems of Consumer Demand Functions." *Journal* *of Business and Economic Statistics* 6: 303-12.

Färe, R. and D. Primont. 1995. *Multi-Output Production and Duality: Theory and Applications*. Boston: Kluwer-Nijhoff Publishing.

Freund, R.J. 1956. "The Introduction of Risk into a Programming Model." *Econometrica,* 24: 253-263.

Gardner, B.L. and R.A. Kramer. 1986. "Experience with Crop Insurance Programs in the United States." *Crop Insurance for Agricultural Development: Issues and Experience.* P. Hazell, C. Pomereda, A. Valdes, eds. Baltimore: Johns Hopkins University Press.

Goldman, S.M. and H. Uzawa. 1964. "A Note on Separability in Demand Analysis." *Econometrica* 32: 387-98.

Goodwin, B.K., V.H. Smith and C. Hammond. 1999. "An Ex-Post Evaluation of the Conservation Reserve Program: Participation, Erosion, and Interaction with Related Programs." Unpublished Manuscript.

Gorman, W.M. 1953. "Community Preference Fields." *Econometrica* 21: 63-80.

_____. 1961. "On a Class of Preference Fields." *Metroeconomica* 13: 53-56.

_____. 1981. "Some Engel Curves." In A. Deaton, ed. *Essays in Honour of Sir Richard Stone*, Cambridge: Cambridge University Press.

Hall, R.E. 1973. "The Specification of Technology with Several Kinds of Output." *Journal of Political Economy* 81: 878-92.

Harwood, J., R. Heifner, K. H. Coble, J. Perry, and A. Somwaru. 1999. "Managing Risk in Farming: Concepts, Research and Analysis." Technical Report, Economic Research Service, U.S. Department of Agriculture.

Hausman, J. 1981. "Exact Consumer's Surplus and Deadweight Loss." *American Economic Review* 71: 662-76.

Hermann, 1975. R. *Lie Groups: History, Frontiers, and Applications, Volume I. Sophus Lie's 1880 Transformation Group Paper*. Brookline MA: Math Sci Press.

Hildenbrand, W. 2007. "Aggregation Theory." Entry for *The New Palgrave of Economics, 2nd Edition*.

Horowitz, J. and E. Lichtenberg. 1994. "Risk Reducing and Risk Increasing Effects of Pesticides." *Journal of Agricultural Economics.* 45: 82-89.

Howe, H., R.A. Pollak, and T.J. Wales. 1979. “Theory and Time Series Estimation of the Quadratic Expenditure System.” *Econometrica* 47:1231-47.

Hueth, D. L. and W. H. Furtan, eds. 1994. *Economics of Agricultural Crop Insurance: Theory and Evidence* New York: Kluwer Academic Publishers.

Huizinga, J. 1993. “Inflation Uncertainty, Relative Price Uncertainty, and Investment in U.S. Manufacturing.” *Journal of Money, Credit, and Banking* 25: 521-49.

Jerison, M. 1993. “Russell on Gorman's Engel Curves: A Correction.” *Economics Letters* 23:171-75.

Jorgenson, D.W., L.J. Lau, and T.M. Stoker. 1980. “Welfare Comparisons under Exact Aggregation.” *American Economic Review* 70: 268-72.

_____. 1982. “The Transendental Logarithmic Model of Aggregate Consumer Behavior,” in *Advances in Econometrics*, R.L. Basmann and G.F. Rhodes, Jr., eds., Greenwich: JAI Press.

Jorgenson, D.W. and D.T. Slesnick. 1984. “Aggregate Consumer Behavior and the Measurement of Inequality.” *Review of Economic Studies* 51: 369-92.

_____. 1987. “Aggregate Consumer Behavior and Household Equivalence Scales.” *Journal of Business and Economic Statistics* 5: 219-232.

Josephson, G.R., R.B. Lord, and C.W. Mitchell. 2000. “Actuarial Documentation of Multiple Peril Crop Insurance Ratemaking Procedures.” Technical Report, Risk Management Agency.

Just, R.E. 1974. “An Investigation of the Importance of Risk in Farmers’ Decisions.” *American Journal of Agricultural Economics* 56: 14-25.

Just, R.E., L. Calvin, and J. Quiggin. 1999. “Adverse Selection in Crop Insurance: Actuarial and Asymmetric Information Incentives.” *American Journal of Agricultural Economics*, 81: 834–849.

Just, R., Zilberman, D. and E. Hochman. 1983. “Estimation of Multicrop Production Functions.” *American Journal of Agricultural Economics* 65: 770-80.

Keeton, K., J. Skees and J. Long. 1999. “The Potential Influence of Risk Management Programs on Cropping Decisions.” Paper presented at the annual meeting of the American Agricultural Economics Association, Nashville, TN.

Knight, T.O. and K. Coble. 1997. “Survey of U.S. Multiple Peril Crop Insurance Since 1980.” *Review of Agricultural Economics.* 19: 128-156.

Kohli, U. 1983. "Non-Joint Technologies." *Review of Economic Studies* 50: 209-219.

LaFrance, J.T. 2004. "Integrability of the Linear Approximate Almost Ideal Demand System." *Economic Letters* 84: 297-303.

_____. 2008. "The Structure of U.S. Food Demand." *Journal of Econometrics* 147: 336-349.

LaFrance, J.T., T.K.M. Beatty, and R.D. Pope. 2006. "Gorman Engel Curves for Incomplete Demand Systems." In M.T. Holt and J.–P. Chavas, eds., *Exploring Frontiers in Applied Economics: Essays in Honor of Stanley R. Johnson*, BE Press.

LaFrance, J.T., T.K.M. Beatty, R.D. Pope and G.K. Agnew. 2002. "Information Theoretic Measures of the U.S. variable cost Distribution in Food Demand." *Journal of Econometrics* 107:235-57.

LaFrance, J.T. and R.D. Pope. 2008. "Homogeneity and Supply." *American Journal of Agricultural Economics* 92: in press.

_____. 2009. "The Generalized Quadratic Expenditure System." In Daniel Slottje, ed. *Contributions to Economic Analysis: Quantifying Consumer Preferences*, New York: Elsevier-Science, 84-116.

_____. 2010. "Duality Theory for Variable Costs in Joint Production." *American Journal of Agricultural* Economics, 92: in press.

LaFrance, J.T., J.P. Shimshack and S.Y. Wu. "Subsidized Crop Insurance and the Extensive Margin." *USDA Economists' Group*, Washington DC, September 2000. Also at Istituto di Servizi per il Mercato Agricolo Alimentare (the Italian Agricultural Ministry), *Risk Management in Agriculture* (www.sicuragro.it), 2004.

______. "The Environmental Impacts of Subsidized Crop Insurance." *Proceedings of CWERE*, 2001.

______. LaFrance, J.T., J.P. Shimshack, and S.Y. Wu. "Crop Insurance and the Extensive Margin." *World Congress of Environmental and Resource Economists*, AERE/EAERE, 2002.

Lau, L. 1972. "Profit Functions of Technologies with Multiple Input and Output." *Review of Economics and Statistics* 54: 281-9.

_____. 1978. "Applications of Profit functions" in *Production Economics: A Dual Approach to Theory and Applications*, M. Fuss and D. McFadden, eds., Amsterdam: North-Holland, 134-216.

Lewbel, A. 1987. "Characterizing Some Gorman systems that satisfy consistent aggregation." *Econometrica* 55:1451-59.

_____. 1988. "An Exactly Aggregable Trigonometric Engel Curve Demand System." *Econometric Reviews* 2:97-102.

_____. 1989. "A Demand System Rank Theorem." *Econometrica* 57:701-705.

_____. 1990. "Full Rank Demand Systems." *International Economic Review* 31:289-300.

_____. 1991. "The Rank of Demand Systems: Theory and Nonparametric Estimation." *Econometrica* 59:711-30.

Lin, W., G. Dean and C. Moore. "An Empirical Test of Utility vs. Profit Maximization in Agricultural Production," *American Journal of Agricultural Economics* 56 (1974): 497-508.

Lopez, R. 1983. "Structural Implications of a Class of Flexible Functional Forms for Profit Functions." *International Economic Review* 26: 593-601.

Markowitz, H.M. 2000. *Portfolio Selection* New York: Blackwell Publishers.

McLaren, K.R. and K.K.G. Wong. 2007. "Effective Global Regularity and Empirical Modeling of Direct, Inverse, and Mixed Demand Systems." Australasian Meeting of the Econometric Society, Brisbane, Queensland, Australia.

Muellbauer, J. 1975. "Aggregation, variable cost Distribution and Consumer Demand." *Review of Economic Studies* 42:525-43.

_____. 1976. "Community Preferences and the Representative Consumer." *Econometrica* 44:979-99.

Nelson, C.H. and E.T. Loehman. 1987. "Further toward a Theory of Agricultural Insurance." *American Journal of Agricultural Economics.* 69: 523-531.

Nerlove, M. 1958. *The Dynamics of Supply: Estimation of Farmers' Response to Price*. Baltimore MD: Johns Hopkins University Press.

Paxson, C.H. 1992. "Using Weather Variability to Estimate the Response of Savings to Transitory Income in Thailand." *American Economics Review* 82: 15-33.

Pflueger, B.W., and P.J. Barry. 1986. "Crop Insurance and Credit: A Farm Level Simulation Analysis," *Agricultural Finance Review* 46: 1-14.

Pope, R.D. 2003. "Risk and Agriculture." in D.J. Meyer ed., *Economics of Risk*, Kalamazoo MI: Upjohn Institute.

Pope, R.D. and J.-P. Chavas. 1994. "Cost Functions Under Production Uncertainty." *American Journal of Agricultural Economics*. 76: 196-204.

Pope, R., J.-P. Chavas, and R.E. Just. "Economic Welfare Evaluations for Producers Under Uncertainty," *American Journal of Agricultural Economics* 65, l (1983): 98-107.

Pope, R.D., and R.E. Just. 1991. "On Testing the Structure of Risk Preferences in Agricultural Supply Analysis," *American Journal of Agricultural Economics* 73, 3: 743-748.

_____. 1996. "Empirical Implementation of Ex ante Cost Functions." *Journal of Econometrics* 72: 231-49.

Pope, R.D., LaFrance, J.T. and R.E. Just. 2007. "Imperfect Price Deflation in Production Systems." *American Journal of Agricultural Economics* 89 (2007): 738-754.

_____. 2010. "Agricultural Arbitrage and Risk Preferences." *Journal of Econometrics,* in press.

Primont, D. and C. Sawyer. 1993. "Recovering the Production Technology from the Cost Function." *The Journal of Productivity Analysis* 4: 347-52.

Quiggin, J. "Some Observations on Insurance, Bankruptcy, and Input Demand." *Journal of Economic Behavior and Organization.* 18 (1992): 101-110.

Rothchild, M. and J. Stiglitz. 1976. "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information." *Quarterly Journal of Economics.* 90: 629-49.

Russell, T. 1983. "On a Theorem of Gorman." *Economic Letters* 11: 223-24.

_____. 1996. "Gorman Demand Systems and Lie Transformation Groups: A Reply." *Economic Letters* 51: 201-4.

Russell, T. and F. Farris. 1993. "The Geometric Structure of Some Systems of Demand Functions." *Journal of Mathematical Economics* 22: 309-25.

_____. 1998. "Integrability, Gorman Systems, and the Lie Bracket Structure of the Real Line." *Journal of Mathematical Economics* 29: 183-209.

Shumway, C. R. 1983. "Supply, Demand, and Technology in a Multiproduct Industry: Texas Field Crops." *American Journal of Agricultural Economics* 65: 748-60.

Skees, J. R. and M. R. Reed. 1986. "Rate Making for Farm–Level Crop Insurance: Implications for Adverse Selection." *American Journal of Agricultural Economics*, 68: 653–659.

Smith, V.H. and V.K. Goodwin. 1996. “Crop Insurance, Moral Hazard and Agricultural Chemical Use.” *American Journal of Agricultural Economics.* 78: 428-438.

Soule, M., W. Nimon, and D. Mullarkey. 2000. “Risk Management and Environmental Outcomes: Framing the Issues.” Paper presented at Crop Insurance, Land Use and the Environment Workshop, USDA/ERS, Washington DC.

Stohs, S. M. 2003. “A Bayesian Updating Approach to Crop Insurance Ratemaking.” Ph.D. Dissertation, Department of Agricultural and Resource Economics, University of California, Berkeley.

Turvey, C.G. 1992. “An Economic Analysis of Alternative Farm Revenue Insurance Policies.” *Canadian Journal of Agricultural Economics* 40: 403-426.

von Haefen, R.H. 2002. “A Complete Characterization of the Linear, Log-Linear, and Semi-Log Incomplete Demand System Models.” *Journal of Agricultural and Resource Economics*, 27: 281-319.

van Daal, J. and A.H.Q.M. Merkies. 1989. “A Note on the Quadratic Expenditure Model.” *Econometrica* 57: 1439-43.

Wales, T.J. and A.D. Woodland. 1983. “”Estimation of Consumer Demand Systems with Binding Non-Negativity Constraints.” *Journal of Econometrics* 21: 263-85.

Williams, J.R. 1988. “A Stochastic Dominance Analysis of Tillage and Crop Insurance Practices in a Semiarid Region.” *American Journal of Agricultural Economics* 70: 112-120.

Wright, B.D. and J.A. Hewitt. 1994. “All-risk Crop Insurance: Lessons from Theory and Experience.” In D.L. Hueth and W.H. Furtan, eds. *Economics of Agricultural Crop Insurance: Theory and Evidence*. Natural Resource Management and Policy Series. Boston, MA: Kluwer Academic Publishers. Chapter 4:73-112.

Wu, J. 1999. “Crop Insurance, Acreage Decisions and Non-point Source Pollution.” *American Journal of Agricultural Economics* 81: 305-320.

Young, C.E., R.D. Schnepf, J.R. Skees, W.W. Lin. 2000. “Production and Price Impacts of U.S. Crop Insurance Subsidies: Some Preliminary Results.” Paper presented at Crop Insurance, Land Use and the Environment Workshop, USDA/ERS, Washington DC.

Appendix A

Let $\boldsymbol{x} \in \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^{n_x}_{++}$ be an n_x–vector of variable inputs, let $\boldsymbol{w} \in \boldsymbol{\mathcal{W}} \subseteq \mathbb{R}^{n_x}_{++}$ be an n_x–vector of variable input prices, let $\boldsymbol{y} \in \boldsymbol{\mathcal{Y}} \subseteq \mathbb{R}^{n_y}_{++}$ be an n_y–vector of outputs, let $\boldsymbol{z} \in \mathcal{Z} \subseteq \mathbb{R}^{n_z}_{++}$ be an n_z–vector of fixed inputs, let $F : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} \times \mathcal{Z} \to \mathbb{R}$ be a transformation function that defines the boundary of a closed, convex production possibilities set with free disposal in inputs and outputs, let $\boldsymbol{X} : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Y}} \times \mathcal{Z} \to \boldsymbol{\mathcal{X}},$ be an n_x–vector of variable input demand functions, and let $C : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Y}} \times \mathcal{Z} \to \mathbb{R}_{++}$ be a variable cost function,

$$c = C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \min_{\boldsymbol{x}} \left\{ \boldsymbol{w}^{\intercal} \boldsymbol{x} : F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \le 0, \boldsymbol{x} \ge \boldsymbol{0} \right\} \equiv \boldsymbol{w}^{\intercal} \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}), \tag{A.1}$$

where the symbol $^{\intercal}$ denotes vector and matrix transposition. The purpose of this appendix is to prove that short-run cost-minimizing variable input demands, $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$, can be written in the form $\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z})$ if and only if $c = C(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) \Leftrightarrow F(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))$.

The neoclassical model of conditional demands for variable inputs with joint production, fixed inputs, and production uncertainty is

$$\boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) = \arg\min \left\{ \boldsymbol{w}^{\intercal} \boldsymbol{x} : F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \le 0,\ \boldsymbol{x} \ge \boldsymbol{0} \right\}, \tag{A.2}$$

where $\boldsymbol{x}$ is an n_x–vector of positive variable inputs with corresponding positive prices, $\boldsymbol{w}$, $\boldsymbol{y}$ is an n_y–vector of planned outputs, $\boldsymbol{z}$ is an n_z–vector of fixed inputs, F is the real valued transformation function that defines the boundary of a closed, convex production possibilities set with free disposal in the inputs and the outputs, $\boldsymbol{X}$ maps variable input prices, planned outputs, and fixed inputs into variable input demand functions, and $C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \boldsymbol{w}^{\intercal} \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}),$ is the positive-valued variable cost function. By Shephard's Lemma, we have

$$\boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) = \nabla_{\boldsymbol{w}} C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv (\partial C / \partial w_1, \cdots, \partial C / \partial w_{n_x})^{\intercal}. \tag{A.3}$$

$\boldsymbol{X}$ is homogeneous of degree zero in $\boldsymbol{w}$ by the derivative property of homogeneous functions. Integrating with respect to $\boldsymbol{w}$ to recover the variable cost function, we obtain

$$c = C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})), \tag{A.4}$$

where $\theta : \boldsymbol{\mathcal{Y}} \times \mathcal{Z} \to \mathbb{R}$ is the *constant of integration*. In the present case, this means that θ is constant with respect to $\boldsymbol{w}$. In general, θ is a function of $\boldsymbol{y}$ and $\boldsymbol{z}$ and its structure cannot be identified from the variable input demands because it captures that part of the joint production process relating to fixed inputs and outputs that is separable from the variable inputs.

Under standard conditions, the variable cost function is strictly decreasing in $\boldsymbol{z}$, strictly increasing in $\boldsymbol{y}$, jointly convex in $(\boldsymbol{y},\boldsymbol{z})$, increasing, concave and homogeneous of degree one in $\boldsymbol{w}$. We are free to choose the *sign* of θ so that, with no loss of generality, $\partial\tilde{C}/\partial\theta > 0$.

Since $\tilde{C}$ is strictly increasing in θ, a unique inverse exists such that $\theta = \gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c)$, where $\gamma : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Z}} \times \mathbb{R}_+ \to \mathbb{R}$, is the inverse of $\tilde{C}$ with respect to θ. $\gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c)$ is called the *quasi-indirect production transformation function,* analogous to the quasi-indirect utility function of consumer theory (Hausman 1981; Epstein 1982; LaFrance 1985, 1986, 1990, 2004; and LaFrance and Hanemann 1989). For all interior and feasible $(\boldsymbol{y},\boldsymbol{z})$, the function γ is strictly increasing in c, strictly decreasing and quasi-convex in $\boldsymbol{w}$, and positively homogeneous of degree zero in ($\boldsymbol{w}$, c).

The following two identities are simple implications of the inverse function theorem:

$$c \equiv \tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c)); \tag{A.5}$$

and

$$\theta \equiv \gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\theta)). \tag{A.6}$$

This lets one write the conditional demands for the variable inputs as

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}}\tilde{C} \equiv \boldsymbol{G}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c). \tag{A.7}$$

Equation (A.7) gives the rationale for writing the factor demands as a function of c as well as $(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})$. Thus, given the above regularity conditions for *F* and *C*, one can always write the system of factor demands as functions of cost.

Now define the quasi-production transformation function by

$$\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \min_{\boldsymbol{w}\geq\boldsymbol{0}}\left\{\gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}^{\mathsf{T}}\boldsymbol{x})\right\}. \tag{A.8}$$

The terminology *quasi-production transformation function* indicates that $\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ only reveals part of the structure of the joint production process. It cannot, and does not, reveal $\theta(\boldsymbol{y},\boldsymbol{z})$. This is analogous to the situation where one only recovers part of a direct utility function when analyzing the market demands for a subset of consumption goods.

The identity $\theta(\boldsymbol{y},\boldsymbol{z}) \equiv \gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})))$ implies

$$\theta(\boldsymbol{y},\boldsymbol{z}) \equiv \gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))) \geq \min_{\boldsymbol{w}\geq\boldsymbol{0}}\left\{\gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}^{\top}\boldsymbol{x})\right\} \equiv \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}), \qquad \text{(A.9)}$$

for all interior and feasible $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$. This inequality follows from the fact that $\theta(\boldsymbol{y},\boldsymbol{z})$ is feasible but not necessarily optimal in the minimization problem. The part of $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ not contained in $\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ is given by (Diewert 1975; Epstein 1975; Hausman 1981; and LaFrance and Hanemman 1989),

$$F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})). \qquad \text{(A.10)}$$

The quasi-production transformation function is the unique solution, $\theta = \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$, to the implicit function, $\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta) = 0,$ in other words, $\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})) \equiv 0.$

The function $\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ in (9) conveys full information about the marginal rates of substitution between variable inputs but only partially so for outputs and fixed inputs. This is again analogous to the situation in consumption theory when one analyzes only a subset of the goods purchased and consumed. This can be shown by applying the implicit function theorem to $\tilde{F},$ which gives

$$\begin{aligned}
\nabla_{\boldsymbol{x}}\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= -\frac{\nabla_{\boldsymbol{x}}\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))}{\nabla_{\theta}\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))},\\
\nabla_{\boldsymbol{y}}\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= -\frac{\nabla_{\boldsymbol{y}}\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))}{\nabla_{\theta}\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))},\\
\nabla_{\boldsymbol{z}}\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= -\frac{\nabla_{\boldsymbol{z}}\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))}{\nabla_{\theta}\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))}.
\end{aligned} \qquad \text{(A.11)}$$

This demonstrates that υ conveys full information on marginal rates of substitution between variable inputs,

$$\frac{\partial\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial x_i}{\partial\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial x_j} = \frac{\partial\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))/\partial x_i}{\partial\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}))/\partial x_j} = \frac{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial x_i}{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial x_j},\ \forall\ i,j = 1,\cdots,n_x, \qquad \text{(A.12)}$$

but only partial information on marginal rates of product transformation between outputs,

$$\begin{aligned}
\frac{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_j} &= \frac{\partial\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial y_i + \partial\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial\theta\cdot\partial\theta(\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial y_j + \partial\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial\theta\cdot\partial\theta(\boldsymbol{y},\boldsymbol{z})/\partial y_j}\\
&\neq \frac{\partial\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_j},\ \forall\ i,j = 1,\cdots,n_y,
\end{aligned} \qquad \text{(A.13)}$$

and marginal rates of substitution between fixed inputs,

$$\frac{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial z_i}{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial z_j} = \frac{\partial \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial z_i + \partial \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial \theta \cdot \partial \theta(\boldsymbol{y},\boldsymbol{z})/\partial z_i}{\partial \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial z_j + \partial \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))/\partial \theta \cdot \partial \theta(\boldsymbol{y},\boldsymbol{z})/\partial z_j}$$
$$\neq \frac{\partial \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial z_i}{\partial \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial z_j}, \ \forall\ i,\ j = 1,\cdots,n_z. \tag{A.14}$$

This background leads directly to the following result.

***Proposition* 1:** *The following functional structures are equivalent:*

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z}); \tag{A.15}$$

$$c = C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})); \tag{A.16}$$

and

$$0 = F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})). \tag{A.17}$$

Proof: (A.16) $\Rightarrow$ (A.15). Differentiating (A.16) with respect to $\boldsymbol{w}$, Shephard’s Lemma implies,

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C}. \tag{A.18}$$

$\tilde{C}$ is strictly monotonic in and has a unique inverse with respect to θ, say $\theta = \tilde{\gamma}(\boldsymbol{w},\boldsymbol{z},c)$. Substituting this into (A.18) obtains

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C}(\boldsymbol{w},\boldsymbol{z},\tilde{\gamma}(\boldsymbol{w},\boldsymbol{z},c)) \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z}). \tag{A.19}$$

(A.17) $\Rightarrow$ (A.15) $\Rightarrow$ (A.16). If the representation of technology has the separable structure in (A.17), then

$$\arg\min\left\{\boldsymbol{w}^{\top}\boldsymbol{x} : \tilde{F}(\boldsymbol{x},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})) \le 0,\ \boldsymbol{x} \ge \boldsymbol{0}\right\} \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})). \tag{A.20}$$

This implies that the variable cost function has the separable structure

$$\boldsymbol{w}^{\top}\tilde{\boldsymbol{X}}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})) \equiv \tilde{C}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})). \tag{A.21}$$

(A.16) $\Rightarrow$ (A.17). Given (A.16), the quasi-production transformation function satisfies

$$\tilde{\upsilon}(\boldsymbol{x},\boldsymbol{z}) \equiv \min_{\boldsymbol{w}\ge\boldsymbol{0}}\left\{\tilde{\gamma}(\boldsymbol{w},\boldsymbol{z},\boldsymbol{w}^{\top}\boldsymbol{x})\right\}. \tag{A.22}$$

This implies that

$$\theta(\boldsymbol{y},\boldsymbol{z}) \equiv \gamma(\boldsymbol{x},\boldsymbol{z},\tilde{C}(\boldsymbol{x},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))) \geq \tilde{\upsilon}(\boldsymbol{x},\boldsymbol{z}), \tag{A.23}$$

for all interior, feasible $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Z}}$, with the boundary of the closed and convex production possibilities set defined by equality on the far right. Since $\tilde{\upsilon}$ is independent of $\boldsymbol{y}$, equations (A.11) and (A.13) imply

$$\frac{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_j} = \frac{\partial \theta(\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial \theta(\boldsymbol{y},\boldsymbol{z})/\partial y_j}, \ \forall\ i, j = 1,\cdots,n_y. \tag{A.24}$$

Hence, the marginal rates of transformation between outputs are independent of variable inputs,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})/\partial y_j}\right) = \frac{\partial}{\partial x_k}\left(\frac{\partial \theta(\boldsymbol{y},\boldsymbol{z})/\partial y_i}{\partial \theta(\boldsymbol{y},\boldsymbol{z})/\partial y_j}\right) = 0, \ \forall\ i, j = 1,\cdots,n_y, \ \forall\ k = 1,\cdots,n_x, \tag{A.25}$$

Thus, $\boldsymbol{y}$ is separable from $\boldsymbol{x}$ in the joint production transformation function (Goldman and Uzawa 1964, Lemma 1), that is, $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \tilde{F}(\boldsymbol{x},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))$. ■

Appendix B

Sufficiency Algebra for Proposition 2

Define the function $\omega:\mathbb{R}_{+}\times\mathbb{R}\to\mathbb{R}$ by

$$\omega(x,y)=y+\int_0^x[\lambda(s)+\omega(s,y)^2]ds, \tag{B.1}$$

where $\lambda:\mathbb{R}\to\mathbb{R}$ is an arbitrary smooth function and w is subject to the pair of initial conditions, $w(0,y)=y$ and $\partial w(0,y)/\partial x=y^2$, to ensure that the definition is unique and smooth. Given two arbitrary smooth functions $\eta:\mathbb{R}_{++}^{n_x}\to\mathbb{R}_{+}$ and $\theta:\mathbb{R}^{n_z}\times\mathbb{R}^{n_y}\to\mathbb{R}$, by Leibniz' Rule of differentiation, we have

$$\frac{\partial\omega(\eta(\boldsymbol{w}),\theta(\boldsymbol{z},\bar{\boldsymbol{y}}))}{\partial\boldsymbol{w}}=\left[\lambda(\eta(\boldsymbol{w}))+\omega(\eta(\boldsymbol{w}),\theta(\boldsymbol{z},\bar{\boldsymbol{y}}))^2\right]\frac{\partial\eta(\boldsymbol{w})}{\partial\boldsymbol{w}}. \tag{B.2}$$

Given a monotonic, smooth function $f:\mathbb{R}_{++}\to\mathbb{C}$, $f'\neq 0$, define the relationship between f and ω by $f=(\alpha\omega+\beta)/(\gamma\omega+\delta)$, $\alpha,\beta,\gamma,\delta:\mathbb{R}_{++}^{n_x}\to\mathbb{C}$, and $\alpha\delta-\beta\gamma\equiv 1$. Let the cost function be $c:\mathbb{R}_{++}^{n_x}\times\mathbb{R}_{++}^{n_z}\times\mathbb{R}_{++}^{n_q}\to\mathbb{R}_{++}$ and denote an arbitrary positive-valued, 1° homogeneous, increasing, and concave deflator by $\pi:\mathbb{R}_{++}^{n_x}\to\mathbb{R}_{++}$. The projective transformation group representation of any exactly aggregable ex ante cost function is

$$f\left(\frac{c(\boldsymbol{w},\boldsymbol{z},\bar{\boldsymbol{y}})}{\pi(\boldsymbol{w})}\right)=\frac{\alpha(\boldsymbol{w})\omega(\eta(\boldsymbol{w}),\theta(\boldsymbol{z},\bar{\boldsymbol{y}}))+\beta(\boldsymbol{w})}{\gamma(\boldsymbol{w})\omega(\eta(\boldsymbol{w}),\theta(\boldsymbol{z},\bar{\boldsymbol{y}}))+\delta(\boldsymbol{w})}. \tag{B.3}$$

Hereafter, suppress all arguments of all functions and use bold italics subscripts to denote vector-valued partial derivatives. For example, rewrite (B.2) compactly as $\omega_{\boldsymbol{w}}=(\lambda+\omega^2)\eta_{\boldsymbol{w}}$.

The inverse of (B.3) with respect to ωw is $\omega=(\delta f-\beta)/(-\gamma f+\alpha)$. Combine this with the identification normalization $\alpha\delta-\beta\gamma\equiv 1$ to obtain the following:

$$\gamma\omega+\delta=\gamma\left(\frac{\delta f-\beta}{-\gamma f+\alpha}\right)+\delta=\frac{\gamma\delta f-\beta\gamma-\gamma\delta f+\alpha\delta}{-\gamma f+\alpha}=\frac{1}{-\gamma f+\alpha}, \tag{B.4}$$

or equivalently, $-\gamma f+\alpha=1/(\gamma\omega+\delta)$. Multiply each side of this by the corresponding side of equation (B.3) to obtain $(-\gamma f+\alpha)f=(\alpha\omega+\beta)/(\gamma\omega+\delta)^2$. These relationships are used in what follows to simplify expressions.

Our task is to differentiate (B.3) with respect to ***w***, combine terms, and rewrite the expression that results so that the elements of $\{1, f, f^2\}$ appear on the right. Differentiating gives

$$f' \cdot \left(\frac{c_w}{\pi} - \frac{c\pi_w}{\pi^2} \right) = \frac{(\alpha_w \omega + \alpha \omega_w + \beta_w)}{(\gamma w + \delta)} - \frac{(\alpha\omega + \beta)(\gamma_w \omega + \gamma \omega_w + \delta_w)}{(\gamma w + \delta)^2}$$
$$= (-\gamma f + \alpha)[\alpha_w \omega + \alpha(\lambda + \omega^2)\eta_w + \beta_w] - (-\gamma f + \alpha) f [\gamma_w \omega + \gamma(\lambda + \omega^2)\eta_w + \delta_w]. \quad \text{(B.5)}$$

The second line follows from $1/(\gamma\omega + \delta) = -\gamma f + \alpha$, $(\alpha\omega + \beta)/(\gamma\omega + \delta)^2 = (-\gamma f + \alpha) f$, and $\omega_w = (\lambda + \omega^2)\eta_w$. Group terms in ω on the second line of (B.5) to obtain

$$f' \cdot \left(\frac{c_w}{\pi} - \frac{c\pi_w}{\pi^2} \right) = (-\gamma f + \alpha)\left[\beta_w + \alpha\lambda\eta_w - (\delta_w + \gamma\lambda\eta_w) f\right]$$
$$+ (-\gamma f + \alpha)(\alpha_w - \gamma_w f)\omega + (-\gamma f + \alpha)^2 \eta_w \omega^2. \quad \text{(B.6)}$$

Substituting $\omega = (\delta f - \beta)/(-\gamma f + \alpha)$ into the second line of (B.6) now leads to

$$f' \cdot \left(\frac{c_w}{\pi} - \frac{c\pi_w}{\pi^2} \right) = (-\gamma f + \alpha)\left[\beta_w + \alpha\lambda\eta_w - (\delta_w + \gamma\lambda\eta_w) f\right]$$
$$+ (-\gamma f + \alpha)(\alpha_w - \gamma_w f)\left(\frac{\delta f - \beta}{-\gamma f + \alpha} \right) + (-\gamma f + \alpha)^2 \eta_w \left(\frac{\delta f - \beta}{-\gamma f + \alpha} \right)^2 \quad \text{(B.7)}$$
$$= (-\gamma f + \alpha)\left[\beta_w + \alpha\lambda\eta_w - (\delta_w + \gamma\lambda\eta_w) f\right] + (\alpha_w - \gamma_w f)(\delta f - \beta) + \eta_w (\delta f - \beta)^2.$$

Expanding the quadratic forms and grouping terms in *f* in the last line of (B.7) gives

$$
\begin{aligned}
f'\cdot\left(\frac{c_w}{\pi}-\frac{c\pi_w}{\pi^2}\right) = & -\gamma f\left[\beta_w+\alpha\lambda\eta_w-(\delta_w+\gamma\lambda\eta_w)f\right] \\
& +\alpha\left[\beta_w+\alpha\lambda\eta_w-(\delta_w+\gamma\lambda\eta_w)f\right] \\
& -\alpha_w\beta+(\alpha_w\delta+\gamma_w\beta)f-\gamma_w\delta f^2 \\
& +\eta_w(\beta^2-2\beta\delta f+\delta^2 f^2) \\
= & \ \alpha(\beta_w+\alpha\lambda\eta_w)-(\gamma\beta_w+\alpha\delta_w+2\alpha\gamma\lambda\eta_w)f+\gamma(\delta_w+\gamma\lambda\eta_v)f^2 \\
& -\alpha_w\beta+\eta_w\beta^2+(\alpha_w\delta+\gamma_w\beta-2\beta\delta\eta_w)f+(-\gamma_w\delta+\eta_w\delta^2)f^2 \\
= & \ \alpha\beta_w-\beta\alpha_w+(\alpha^2\lambda+\beta^2)\eta_w \\
& -\left[\alpha\delta_w-\delta\alpha_w+\gamma\beta_w-\beta\gamma_w+2(\alpha\gamma\lambda+\beta\delta)\eta_w\right]f \\
& +\left[\gamma\delta_w-\delta\gamma_w+(\gamma^2\lambda+\delta^2)\eta_w\right]f^2.
\end{aligned}
\tag{B.8}
$$

Grouping terms in η_w as well gives

$$
\begin{aligned}
f'\cdot\left(\frac{c_w}{\pi}-\frac{c\pi_w}{\pi^2}\right) = & \ \alpha\beta_w-\beta\alpha_w-(\alpha\delta_w-\delta\alpha_w+\gamma\beta_w-\beta\gamma_w)f+(\gamma\delta_w-\delta\gamma_w)f^2 \\
& +\left[(\delta f-\beta)^2+\lambda(-\gamma f+\alpha)^2\right]\eta_w.
\end{aligned}
\tag{B.9}
$$

Finally, solving for $c_w = \boldsymbol{x}$ gives

$$
\begin{aligned}
\boldsymbol{x} = & \ \frac{\pi_w}{\pi}c+\frac{\pi}{f'}\Big\{\alpha\beta_w-\beta\alpha_w-(\alpha\delta_w-\delta\alpha_w+\gamma\beta_w-\beta\gamma_w)f+(\gamma\delta_w-\delta\gamma_w)f^2 \\
& +\left[(\delta f-\beta)^2+(-\gamma f+\alpha)^2\lambda\right]\eta_w\Big\}.
\end{aligned}
\tag{B.10}
$$

Appendix C

Specifying the Cost Function

The first $n_x - 1$ variable input prices, $\boldsymbol{w}$, and total variable cost, c, are normalized by the average wage rate for hired farm labor, w_{n_x}. We consider the following transformation of normalized variable cost, which nests the PIGLOG and PIGL class of models,

$$f(c) = (c^{\kappa} + \kappa - 1)/\kappa, \quad f'(c) = c^{\kappa}, \quad f''(c) = (\kappa - 1)c^{\kappa-2}, \;\; \kappa \in \mathbb{R}_{+}.$$

This includes all of the real-valued Gorman functional forms, with $f(c) = c$ when $\kappa = 1$, and $\lim_{\kappa \to 0} f(c) = 1 + \ln c$. Therefore, the highest rank that the variable input demands can achieve is three (Gorman 1981; Lewbel 1987; LaFrance and Pope 2009).

Previous empirical work considered translated Box-Cox functions of input prices, $(w_i^{\lambda} + \lambda - 1)/\lambda$, $\lambda \in [0,1]$, $i = 1, \cdots, n_x - 1$, to nest models with that have log prices, power functions of prices, and are linear prices. In the national model $\lambda = 1$ is optimal on this interval and for our data set. Hence, we restrict attention here to normalized input prices. Our previous empirical results using this data at state- and national-levels of aggregation and various levels of aggregation across inputs, suggests quite strongly that rank three over-parameterizes this data set (Ball, et al., 2010). Hence, we focus here on rank two:

$$\begin{aligned}
& f(c(\boldsymbol{w}, A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}})) = \alpha(\boldsymbol{w}, A, K) + \beta(\boldsymbol{w})\theta(A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}}), \\
& \Leftrightarrow \tilde{\theta}(\boldsymbol{w}, c, A, K) = \frac{f(c) - \alpha(\boldsymbol{w}, A, K)}{\beta(\boldsymbol{w})}, \\
& \alpha(\boldsymbol{w}, A, K) = (\alpha_{10} + \boldsymbol{\alpha}_1^{\top}\boldsymbol{w})A + (\alpha_{20} + \boldsymbol{\alpha}_2^{\top}\boldsymbol{w})K, \qquad \text{(C.1)} \\
& \beta(\boldsymbol{w}) = \sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w} + 2\boldsymbol{\gamma}^{\top}\boldsymbol{w} + 1}, \\
& \bar{\boldsymbol{Y}} = \bar{\boldsymbol{y}} \bullet \boldsymbol{a}_t = [\bar{y}_1 a_1 \cdots \bar{y}_{n_y} a_{n_y}]^{\top},
\end{aligned}$$

where $\bar{y}_i$ is the expected (planned) yield for the i^{th} crop, a_i is the acreage planted to this crop, and the symbol $\bullet$ denotes the Hadamard/Schur product for matrices and vectors. This appendix identifies restrictions on the parameters in (C.1) that are necessary and sufficient for economic regularity of the variable cost function.

Monotonicity in $\boldsymbol{w}$*:*

$$c^{\kappa-1}\frac{\partial c}{\partial \boldsymbol{w}} = \boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K + \frac{\theta}{\beta}(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma}) \geq \boldsymbol{0}$$
$$\Leftrightarrow \tilde{\boldsymbol{x}} = c^{1-\kappa}\left[\boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K + \left(\frac{f-\alpha}{\beta^2}\right)(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})\right] \geq \boldsymbol{0}, \tag{C.2}$$

where $\tilde{\boldsymbol{x}} = [x_1 \cdots x_{n_x-1}]^\top$ is the (n_x–1)–vector of the first n_x–1 input quantities, excluding labor.

Concavity in ***w****:*

$$(\kappa-1)c^{\kappa-2}\frac{\partial c}{\partial \boldsymbol{w}}\frac{\partial c}{\partial \boldsymbol{w}^\top} + c^{\kappa-1}\frac{\partial^2 c}{\partial \boldsymbol{w}\partial \boldsymbol{w}^\top} = \frac{\theta}{\beta}\boldsymbol{B} - \frac{\theta}{\beta^3}(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})^\top$$
$$\Leftrightarrow \frac{\partial^2 c}{\partial \boldsymbol{w}\partial \boldsymbol{w}^\top} = \left(\frac{1-\kappa}{c}\right)\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^\top + c^{1-\kappa}\left(\frac{f-\alpha}{\beta^2}\right)\left[\boldsymbol{B} - \frac{(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})^\top}{\beta^2}\right], \tag{C.3}$$

The first matrix on the right-hand-side of the second line is rank and is negative semi-definite if and only if $\kappa \geq 1$. The matrix in square brackets on the far right of the second line will be positive semi-definite if $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^\top + \boldsymbol{\gamma}\boldsymbol{\gamma}^\top$, where $\boldsymbol{L}$ is a triangular matrix with nonzero main diagonal elements. This makes the following $n_x \times n_x$ matrix positive definite:

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^\top & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & \boldsymbol{\gamma} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{L}^\top & \boldsymbol{0}^\top \\ \boldsymbol{\gamma}^\top & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}\boldsymbol{L}^\top + \boldsymbol{\gamma}\boldsymbol{\gamma}^\top & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^\top & 1 \end{bmatrix}, \tag{C.4}$$

since it give a Choleski factorization of the matrix on the left. It follows from this that $\boldsymbol{w}^\top\boldsymbol{B}\boldsymbol{w} + 2\boldsymbol{\gamma}^\top\boldsymbol{w} + 1 = [\boldsymbol{w}^\top\, 1]\begin{bmatrix} \boldsymbol{B} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^\top & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ 1 \end{bmatrix} > 0 \;\forall\; \boldsymbol{w} \in \mathbb{R}^{n_x-1}_{++}$, and $\left[\boldsymbol{B} - \frac{(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})(\boldsymbol{B}\boldsymbol{w}+\boldsymbol{\gamma})^\top}{\boldsymbol{w}^\top\boldsymbol{B}\boldsymbol{w} + 2\boldsymbol{\gamma}^\top\boldsymbol{w} + 1}\right]$ is positive semi-definite, by the Cauchy-Schwartz inequality in n_x–dimensional Euclidean space. Given this, the second term on the right-hand-side of the second line of equation (C.3) will be negative semi-definite if and only if $f \leq \alpha$.

Constant returns to scale (*CRS*):

$$\theta \equiv \frac{\partial \theta}{\partial A}A + \frac{\partial \theta}{\partial K}K + \frac{\partial \theta}{\partial \boldsymbol{a}^\top}\boldsymbol{a} + \frac{\partial \theta}{\partial \bar{\boldsymbol{Y}}^\top}\bar{\boldsymbol{Y}}. \tag{C.5}$$

We believe that we have a much more accurate measure of capital than we do of land. Hence, we normalize θ by the value of capital rather than land in farms.

Monotonicity in $(A, K, \boldsymbol{a}, \overline{\boldsymbol{Y}})$:

$$c^{\kappa-1}\frac{\partial c}{\partial A} = \alpha_{10} + \boldsymbol{\alpha}_1^{\top}\boldsymbol{w} + \beta\frac{\partial \theta}{\partial A} \le 0, c^{\kappa-1}\frac{\partial c}{\partial K} = \alpha_{20} + \boldsymbol{\alpha}_2^{\top}\boldsymbol{w} + \beta\frac{\partial \theta}{\partial K} \le 0,$$
$$c^{\kappa-1}\frac{\partial c}{\partial \boldsymbol{a}} = \beta\frac{\partial \theta}{\partial \boldsymbol{a}} \le \mathbf{0}, c^{\kappa-1}\frac{\partial c}{\partial \overline{\boldsymbol{Y}}} = \beta\frac{\partial \theta}{\partial \overline{\boldsymbol{Y}}} \le \mathbf{0}. \tag{C.6}$$

Joint Convexity in $(A, K, \boldsymbol{a}, \overline{\boldsymbol{Y}})$:

$$\begin{bmatrix} \frac{\partial^2 c}{\partial A^2} & \frac{\partial^2 c}{\partial A \partial K} & \frac{\partial^2 c}{\partial A \partial \boldsymbol{a}^{\top}} & \frac{\partial^2 c}{\partial A \partial \overline{\boldsymbol{Y}}^{\top}} \\ \frac{\partial^2 c}{\partial K \partial A} & \frac{\partial^2 c}{\partial K^2} & \frac{\partial^2 c}{\partial K \partial \boldsymbol{a}^{\top}} & \frac{\partial^2 c}{\partial K \partial \overline{\boldsymbol{Y}}^{\top}} \\ \frac{\partial^2 c}{\partial \boldsymbol{a} \partial A} & \frac{\partial^2 c}{\partial \boldsymbol{a} \partial K} & \frac{\partial^2 c}{\partial \boldsymbol{a} \partial \boldsymbol{a}^{\top}} & \frac{\partial^2 c}{\partial \boldsymbol{a} \partial \overline{\boldsymbol{Y}}^{\top}} \\ \frac{\partial^2 c}{\partial \overline{\boldsymbol{Y}} \partial A} & \frac{\partial^2 c}{\partial K \partial \overline{\boldsymbol{Y}}} & \frac{\partial^2 c}{\partial \boldsymbol{a} \partial \overline{\boldsymbol{Y}}} & \frac{\partial^2 c}{\partial \overline{\boldsymbol{Y}} \partial \overline{\boldsymbol{Y}}^{\top}} \end{bmatrix} = \frac{1-\kappa}{c} \begin{bmatrix} \frac{\partial c}{\partial A} \\ \frac{\partial c}{\partial K} \\ \frac{\partial c}{\partial \boldsymbol{a}} \\ \frac{\partial c}{\partial \overline{\boldsymbol{Y}}} \end{bmatrix} \begin{bmatrix} \frac{\partial c}{\partial A} \\ \frac{\partial c}{\partial K} \\ \frac{\partial c}{\partial \boldsymbol{a}} \\ \frac{\partial c}{\partial \overline{\boldsymbol{Y}}} \end{bmatrix}^{\top} + c^{1-\kappa}\beta \begin{bmatrix} \frac{\partial^2 \theta}{\partial A^2} & \frac{\partial^2 \theta}{\partial A \partial K} & \frac{\partial^2 \theta}{\partial A \partial \boldsymbol{a}^{\top}} & \frac{\partial^2 \theta}{\partial A \partial \overline{\boldsymbol{Y}}^{\top}} \\ \frac{\partial^2 \theta}{\partial K \partial A} & \frac{\partial^2 \theta}{\partial K^2} & \frac{\partial^2 \theta}{\partial K \partial \boldsymbol{a}^{\top}} & \frac{\partial^2 \theta}{\partial K \partial \overline{\boldsymbol{Y}}^{\top}} \\ \frac{\partial^2 \theta}{\partial \boldsymbol{a} \partial A} & \frac{\partial^2 \theta}{\partial \boldsymbol{a} \partial K} & \frac{\partial^2 \theta}{\partial \boldsymbol{a} \partial \boldsymbol{a}^{\top}} & \frac{\partial^2 \theta}{\partial \boldsymbol{a} \partial \overline{\boldsymbol{Y}}^{\top}} \\ \frac{\partial^2 \theta}{\partial \overline{\boldsymbol{Y}} \partial A} & \frac{\partial^2 \theta}{\partial K \partial \overline{\boldsymbol{Y}}} & \frac{\partial^2 \theta}{\partial \boldsymbol{a} \partial \overline{\boldsymbol{Y}}} & \frac{\partial^2 \theta}{\partial \overline{\boldsymbol{Y}} \partial \overline{\boldsymbol{Y}}^{\top}} \end{bmatrix}. \tag{C.7}$$

The first matrix on the right is rank one and will be positive semi-definite if and only if $\kappa \le 1$. Therefore, $c(\boldsymbol{w}, A, K, \boldsymbol{a}, \overline{\boldsymbol{Y}})$ will be concave in $\boldsymbol{w}$ and jointly convex in $(A, K, \boldsymbol{a}, \overline{\boldsymbol{Y}})$ more than locally if and only if $\kappa = 1$. We estimated the rank two model using the Box-Cox transformation on cost. The NL3SLS/GMM point estimate for κ is 1.124 with a classical (Gaussian) asymptotic standard error of .152 and a White/Huber heteroskedasticity consistent standard error of .111. We can not reject a null hypothesis of $\kappa = 1$ in either case at the 25% significance level. Hence, in this paper we restrict our attention to $\kappa = 1$.

Given this restriction, the cost function will be jointly convex in $(A, K, \boldsymbol{a}, \overline{\boldsymbol{Y}})$ if and only if the Hessian matrix for θ,

$$\begin{bmatrix} \frac{\partial^2\theta}{\partial A^2} & \frac{\partial^2\theta}{\partial A\partial K} & \frac{\partial^2\theta}{\partial A\partial \boldsymbol{a}^\top} & \frac{\partial^2\theta}{\partial A\partial \bar{\boldsymbol{Y}}^\top} \\ \frac{\partial^2\theta}{\partial K\partial A} & \frac{\partial^2\theta}{\partial K^2} & \frac{\partial^2\theta}{\partial K\partial \boldsymbol{a}^\top} & \frac{\partial^2\theta}{\partial K\partial \bar{\boldsymbol{Y}}^\top} \\ \frac{\partial^2\theta}{\partial \boldsymbol{a}\partial A} & \frac{\partial^2\theta}{\partial \boldsymbol{a}\partial K} & \frac{\partial^2\theta}{\partial \boldsymbol{a}\partial \boldsymbol{a}^\top} & \frac{\partial^2\theta}{\partial \boldsymbol{a}\partial \bar{\boldsymbol{Y}}^\top} \\ \frac{\partial^2\theta}{\partial \bar{\boldsymbol{Y}}\partial A} & \frac{\partial^2\theta}{\partial K\partial \bar{\boldsymbol{Y}}} & \frac{\partial^2\theta}{\partial \boldsymbol{a}\partial \bar{\boldsymbol{Y}}} & \frac{\partial^2\theta}{\partial \bar{\boldsymbol{Y}}\partial \bar{\boldsymbol{Y}}^\top} \end{bmatrix}, \tag{C.8}$$

is positive semi-definite. Given these considerations, the specification for θ employed in the paper is

$$\theta(A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t) = -\theta_1 A_t - \theta_2 K_t - \boldsymbol{\theta}_3^\top \boldsymbol{a}_t + \boldsymbol{\theta}_4^\top \bar{\boldsymbol{Y}}_t + ½\left(\frac{\theta_5 A_t^2 + \boldsymbol{a}_t^\top \boldsymbol{\Theta}_6 \boldsymbol{a}_t + \bar{\boldsymbol{Y}}_t^\top \boldsymbol{\Theta}_7 \bar{\boldsymbol{Y}}_t}{K_t} \right), \tag{C.9}$$

where $\theta_1, \theta_2, \theta_5 > 0$, $\boldsymbol{\theta}_3, \boldsymbol{\theta}_4 > \mathbf{0}$, and $\boldsymbol{\Theta}_6, \boldsymbol{\Theta}_7$ are symmetric and positive semidefinite. The implied constraints for monotonicity can be written as

$$\begin{aligned} &\frac{\partial c_t}{\partial A_t} < 0 \ \forall\ t \Leftrightarrow \min_t \left(\theta_1 - \theta_5 \frac{A_t}{K_t} \right) > \max_t \left(\frac{\alpha_0 + \boldsymbol{\alpha}_1^\top \boldsymbol{w}_t}{\sqrt{\beta(\boldsymbol{w})}} \right), \\ &\frac{\partial c_t}{\partial K_t} < 0 \ \forall\ t \Leftrightarrow \min_t \left[\theta_2 + ½\left(\frac{\theta_5 A_t^2 + \boldsymbol{a}_t^\top \boldsymbol{\Theta}_6 \boldsymbol{a}_6 + \bar{\boldsymbol{Y}}_t^\top \boldsymbol{\Theta}_7 \bar{\boldsymbol{Y}}}{K_t^2} \right) \right] > \max_t \left(\frac{\alpha_2 + \boldsymbol{\alpha}_3^\top \boldsymbol{w}_t}{\sqrt{\beta(\boldsymbol{w})}} \right), \\ &\frac{\partial c_t}{\partial \boldsymbol{a}_t} < \mathbf{0} \ \forall\ t \Leftrightarrow \boldsymbol{\theta}_3 > \max_t \left(\frac{\boldsymbol{\Theta}_6 \boldsymbol{a}_6}{K_t} \right), \\ &\frac{\partial c_t}{\partial \bar{\boldsymbol{Y}}_t} > \mathbf{0} \ \forall\ t \Leftrightarrow \boldsymbol{\theta}_4 + \min_t \left(\frac{\boldsymbol{\Theta}_7 \bar{\boldsymbol{Y}}_t}{K_t} \right) > \mathbf{0}. \end{aligned} \tag{C.10}$$

These can be imposed iteratively in estimation, if necessary (LaFrance 1991). In this paper, we checked for the monotonicity conditions at each data point given the parameter estimates obtained without imposing monotonicity.

Also, given that $K_t > 0$, the implied curvature conditions are that the matrix

$$\begin{bmatrix} \theta_5 K_t^2 & -\theta_5 A_t K_t & \mathbf{0}_{n_y}^\top & \mathbf{0}_{n_y}^\top \\ -\theta_5 A_t K_t & (\theta_5 A_t^2 + \boldsymbol{a}_t^\top \boldsymbol{\Theta}_6 \boldsymbol{a}_6 + \bar{\boldsymbol{Y}}_t^\top \boldsymbol{\Theta}_7 \bar{\boldsymbol{Y}}_t) & -K_t \boldsymbol{a}_t^\top \boldsymbol{\Theta}_6 & -K_t \bar{\boldsymbol{Y}}_t^\top \boldsymbol{\Theta}_7 \\ \mathbf{0}_{n_y} & -K_t \boldsymbol{\Theta}_6 \boldsymbol{a}_6 & K_t^2 \boldsymbol{\Theta}_6 & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_y} & -K_t \boldsymbol{\Theta}_7 \bar{\boldsymbol{Y}}_t & \mathbf{0}_{n_y \times n_y} & K_t^2 \boldsymbol{\Theta}_7 \end{bmatrix} \tag{C.11}$$

is positive semidefinite. This can be imposed during estimation with the Choleski factors, $\boldsymbol{\Theta}_6 = \boldsymbol{L}_6 \boldsymbol{L}_6^\top$ and $\boldsymbol{\Theta}_7 = \boldsymbol{L}_7 \boldsymbol{L}_7^\top$, with $\boldsymbol{L}_6$ and $\boldsymbol{L}_7$ lower triangular Choleski factors for $\boldsymbol{\Theta}_6$ and $\boldsymbol{\Theta}_7$, respectively, and the inequality $\theta_5 > 0$. In this paper, only the matrix $\boldsymbol{\Theta}_7 = \boldsymbol{L}_7 \boldsymbol{L}_7^\top$ is estimated as part of the arbitrage conditions.

Appendix D

D.1 Output Specific (Embodied) Technological Change

The specification for θ_t employed in this paper is

$$\theta_t(A_t, K_t, \boldsymbol{a}_t, \bar{\boldsymbol{Y}}_t) = -\theta_1 A_t - \theta_2 K_t - \boldsymbol{\theta}_3^{\top}\boldsymbol{a}_t + \boldsymbol{\theta}_{4,t}^{\top}\bar{\boldsymbol{Y}}_t + ½\left(\frac{\theta_5 A_t^2 + \boldsymbol{a}_t^{\top}\boldsymbol{\Theta}_6\boldsymbol{a}_t + \bar{\boldsymbol{Y}}_t^{\top}\boldsymbol{\Theta}_{7,t}\bar{\boldsymbol{Y}}_t}{K_t}\right), \quad \text{(D.1)}$$

where $\theta_1, \theta_2, \theta_5 > 0$, $\boldsymbol{\theta}_3, \boldsymbol{\theta}_{4,t} > \mathbf{0}$, $\boldsymbol{\Theta}_6$, $\boldsymbol{\Theta}_{7,t}$ are symmetric and positive semi-definite, and $\bar{\boldsymbol{Y}}_t = \bar{\boldsymbol{y}}_t \bullet \boldsymbol{a}_t$ is the vector of planned crop production with $\bar{\boldsymbol{y}}_t$ the vector of expected yields per planted acre and $\boldsymbol{a}_t$ the vector of planted acres. The subscripts t have been added to θ_t, $\boldsymbol{\theta}_{4,t}$, $\boldsymbol{\Theta}_{7,t}$ to indicate that the technology is potentially time-varying as a result of exogenous embodied technological change in yield per acre (e.g., due to the development of hybrid seeds, genetically modified plant organisms, etc.).

We model this relative to $t = 0$ via the following vector-valued equations:

$$\bar{\boldsymbol{y}}_t = \boldsymbol{\phi}(t) \bullet \bar{\boldsymbol{y}}_0,\ \boldsymbol{\phi}(0) = \boldsymbol{\iota},\ \boldsymbol{\phi}'(t) \geq \mathbf{0}\ \forall\ t \geq 0, \quad \text{(D.2)}$$

where $\bullet$ denotes the Hadamard product, i.e., $\boldsymbol{x} \bullet \boldsymbol{y} = [x_1 y_1\ x_2 y_2\ \cdots\ x_n y_n]^{\top}$. Given this, then $\forall\ (A, K, \boldsymbol{a}, \bar{\boldsymbol{y}}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^{n_y} \times \mathbb{R}_+^{n_y}$, the variable cost function will be constant across time periods for any if and only if:

$$\theta_t(A, K, \boldsymbol{a}, \boldsymbol{\phi}(t) \bullet \bar{\boldsymbol{y}} \bullet \boldsymbol{a}) = \theta_0(A, K, \boldsymbol{a}, \bar{\boldsymbol{y}} \bullet \boldsymbol{a}). \quad \text{(D.3)}$$

Applying this to (D.1) gives

$$\begin{gathered} \boldsymbol{\theta}_{4,t}^{\top}\boldsymbol{\phi}(t) \bullet \bar{\boldsymbol{y}} \bullet \boldsymbol{a} = \boldsymbol{\theta}_{4,0}^{\top}\bar{\boldsymbol{y}} \bullet \boldsymbol{a}, \text{ and} \\ [\boldsymbol{\phi}(t) \bullet \bar{\boldsymbol{y}} \bullet \boldsymbol{a}]^{\top}\boldsymbol{\Theta}_{7,t}[\boldsymbol{\phi}(t) \bullet \bar{\boldsymbol{y}} \bullet \boldsymbol{a}] = (\bar{\boldsymbol{y}} \bullet \boldsymbol{a})^{\top}\boldsymbol{\Theta}_{7,0}(\bar{\boldsymbol{y}} \bullet \boldsymbol{a})\ \forall\ (\bar{\boldsymbol{y}} \bullet \boldsymbol{a}) \in \mathbb{R}_+^{n_y} \times \mathbb{R}_+^{n_y}. \end{gathered} \quad \text{(D.4)}$$

Differentiating with respect to $\bar{y}_i a_i$ term by term, then implies that these matrix equations hold if and only if

$$\begin{gathered} \theta_{4,i,t} = \theta_{4,i,0} \big/ \phi_i(t),\ \forall\ i = 1, \cdots, n_y, \text{ and} \\ \Theta_{7,i,j,t} = \Theta_{7,i,j,0} \big/ \phi_i(t)\phi_j(t),\ \forall\ i, j = 1, \cdots, n_y, t = 0, 1, \cdots, T. \end{gathered} \quad \text{(D.5)}$$

or in matrix notation, $\boldsymbol{\theta}_{4,t} = \boldsymbol{\Delta}(\phi_i(t))^{-1}\boldsymbol{\theta}_{4,0}, \boldsymbol{\Theta}_{7,t} = \boldsymbol{\Delta}(\phi_i(t))^{-1}\boldsymbol{\Theta}_{7,0}\boldsymbol{\Delta}(\phi_i(t))^{-1}$, where $\boldsymbol{\Delta}(x_i)$ is a diagonal matrix with i^{th} diagonal element equal to x_i. To illustrate, suppose expected yields follow linear trends, $\boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 + \boldsymbol{\phi}_1 t,\ \boldsymbol{\phi}_0, \boldsymbol{\phi}_1 > \mathbf{0}$. Since both lines of (D.5) are 0° homogeneous in the parameters, normalizations are required for identification. The normalization consistent with $\boldsymbol{\phi}(0) = \boldsymbol{\iota}$ is $\boldsymbol{\phi}_0 = \boldsymbol{\iota}$. This gives the following specification for the impacts of embodied technological change on the cost function as:

$$\begin{aligned} \theta_{4,i,t} &= \theta_{4,i,0} \big/ (1+\phi_i t),\ \forall\ i = 1,\cdots,n_y, \text{ and} \\ \Theta_{7,i,j,t} &= \Theta_{7,i,j,0} \big/ (1+\phi_i t)(1+\phi_j t),\ \forall\ i, j = 1,\cdots,n_y, t = 0,\cdots.T. \end{aligned} \tag{D.6}$$

If the null hypothesis is $H_0 : \boldsymbol{\phi}_1 = \mathbf{0}$, and the alternative is $H_1 : \boldsymbol{\phi}_1 > \mathbf{0}$, then under the alternative, this shifts the system of marginal cost equations downward at the rate of a rectangular hyperbola with an intercept in the denominator, and rotates them downward and to the right as the rate of a system of quadratic translated rectangular hyperbolas. Note that the effects of linear yield trends do not appear linearly anywhere in the dual cost function specification. Hence, linear trends are not likely to appear in output supply equations if this form of technological change is correct.

E.2 Technological Change Specific to Variable Inputs

The variable cost function we have derived for this study is

$$c(\boldsymbol{w}, A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}}) = \boldsymbol{\alpha}_1^{\top}\boldsymbol{w}A + \boldsymbol{\alpha}_2^{\top}\boldsymbol{w}K + \sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w}} \times \theta(A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}}), \tag{D.7}$$

To identify the impact of variable input specific technological change, we need to recover the joint production transformation function. This is relatively straightforward for this specification of cost. The conditional input demands are

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K + \theta \boldsymbol{B} \frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w}}} \\ &= \boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K + \left[\frac{c - \boldsymbol{\alpha}_1^{\top}\boldsymbol{w}A - \boldsymbol{\alpha}_2^{\top}\boldsymbol{w}K}{\sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w}}} \right] \boldsymbol{B} \frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w}}}. \end{aligned} \tag{D.8}$$

Solve the first line of (D.8) for the normalized price vector, $\boldsymbol{w} \big/ \sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w}}$, to obtain

$$\frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^{\top}\boldsymbol{B}\boldsymbol{w}}} = \frac{1}{\theta}\boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K). \tag{D.9}$$

Recall that monotonicity and concavity require $\theta = (c - \boldsymbol{\alpha}_1^\top \boldsymbol{w} A - \boldsymbol{\alpha}_2^\top \boldsymbol{w} K) \big/ \sqrt{\boldsymbol{w}^\top \boldsymbol{B}\boldsymbol{w}} < 0.$ Adding up, $c = \boldsymbol{w}^\top \boldsymbol{x},$ implies

$$\theta = \frac{\boldsymbol{w}^\top \boldsymbol{x} - \boldsymbol{\alpha}_1^\top \boldsymbol{w} A - \boldsymbol{\alpha}_2^\top \boldsymbol{w} K}{\sqrt{\boldsymbol{w}^\top \boldsymbol{B}\boldsymbol{w}}} = \frac{1}{\theta}(\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K)^\top \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K), \tag{D.10}$$

or equivalently,

$$\theta^2 = (\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K)^\top \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K). \tag{D.11}$$

The economically relevant root is the negative one, and the joint production transformation function is

$$\theta(A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}}) + \sqrt{(\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K)^\top \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K)} = 0. \tag{D.12}$$

To model the effects of input specific technological change, define the vector of effective inputs in period t relative to period 0 by

$$\boldsymbol{\varphi}(t) \bullet \boldsymbol{x},\ \boldsymbol{\varphi}(0) = \boldsymbol{\iota},\ \boldsymbol{\varphi}'(t) \geq \boldsymbol{0}\ \forall\ t \geq 0,\ \boldsymbol{\iota} = [1 \cdots 1]^\top, \tag{D.13}$$

With this definition, $\forall\ (\boldsymbol{x}, A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}}) \in \mathbb{R}_+^{n_x} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^{n_y} \times \mathbb{R}_+^{n_y},$ write the period t joint production transformation function as

$$\theta(A, K, \boldsymbol{a}, \bar{\boldsymbol{Y}}) + \sqrt{[\boldsymbol{\varphi}(t) \bullet \boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K]^\top \boldsymbol{B}^{-1}[\boldsymbol{\varphi}(t) \bullet \boldsymbol{x} - \boldsymbol{\alpha}_1 A - \boldsymbol{\alpha}_2 K]} = 0. \tag{D.14}$$

The implication is that if the vector $\boldsymbol{x}$ of quantity units of the variable inputs are purchased and used, then the effective quantity vector is $\boldsymbol{\varphi}(t) \bullet \boldsymbol{x} \geq \boldsymbol{x}\ \forall\ t \geq 0.$ The implied variable cost minimizing vector of input demand equations

$$\begin{aligned}
\boldsymbol{x} &= \boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right](\boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K) + \theta \boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right]\boldsymbol{B}\boldsymbol{\Delta}\left[\varphi_j(t)^{-1}\right]\frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{B}\boldsymbol{w}}} \\
&= \boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right](\boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K) \\
&\quad + \left[\frac{c - (\boldsymbol{\alpha}_1 A + \boldsymbol{\alpha}_2 K)^\top \boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right]\boldsymbol{w}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right]\boldsymbol{B}\boldsymbol{\Delta}\left[\varphi_j(t)^{-1}\right]\boldsymbol{w}}}\right]\frac{\boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right]\boldsymbol{B}\boldsymbol{\Delta}\left[\varphi_j(t)^{-1}\right]\boldsymbol{w}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Delta}\left[\varphi_i(t)^{-1}\right]\boldsymbol{B}\boldsymbol{\Delta}\left[\varphi_j(t)^{-1}\right]\boldsymbol{w}}}.
\end{aligned} \tag{D.15}$$

Similar to output specific technological change, the linear case $\boldsymbol{\varphi}(t) = \boldsymbol{\iota} + \boldsymbol{\varphi}_1 t,\ \boldsymbol{\varphi}_1 \geq \boldsymbol{0},$ implies that technological change shifts and rotates the demand for variable inputs. As a consequence, linear trends in variable input use levels are not likely to be valid specifications for the effects of input specific technological change if this model specification is (approximately) correct.

Appendix E

Specification Errors and Parameter Stability Tests

Many diagnostic procedures for testing parameter stability and model specification errors have been developed. Few are designed for large systems of nonlinear simultaneous equations in small samples. These properties preclude using recursive-forecast residuals or Chow tests based on sequential sample splits to analyze specification errors or non-constant parameters Brown, Durbin, and Evans 1975; Harvey 1990, 1993; Hendry 1995). It is desirable to test whether the data are consistent with the model specification and constant parameters. LaFrance (2008) derived a set of specification and parameter stability diagnostics for this class of problems. These test statistics rely on the estimated in-sample residuals and have power against a range of alternatives, including non-constant parameters and specification errors. The purpose of this section is to discuss briefly the main ideas that underpin this class of test statistics.

If the model is stationary and the errors are innovations, then consistent estimates of the model parameters can be found in any number of ways. Given consistent parameter estimates, the estimated errors converge in probability (and therefore, in distribution) to the true errors, $\hat{\boldsymbol{\varepsilon}}_t \xrightarrow{P} \boldsymbol{\varepsilon}_t$. Therefore, for each $i = 1, \cdots, n_x - 1$, by the central limit theorem for stationary Martingale differences, we have

$$\frac{1}{\sqrt{T}\sigma_i}\sum_{t=1}^{T}\varepsilon_{it} \xrightarrow{D} N(0,1), \tag{1}$$

where $\sigma_i^2 = E(\varepsilon_{it}^2)$ is the variance of the residual for the i^{th} demand equation. Moreover, for any given proportion of the sample, uniformly in $z \in [0,1]$,

$$\frac{1}{\sqrt{T}\sigma_i}\sum_{t=1}^{[zT]}\varepsilon_{it} \xrightarrow{D} N(0,z), \tag{2}$$

where $[zT]$ is the largest integer that does not exceed zT. The variance is z because we sum $[zT]$ independent terms each with variance $1/T$. Multiplying (1) by z and subtracting from (2) then gives

$$\frac{1}{\sqrt{T}\sigma_i}\sum_{t=1}^{[zT]}\left(\varepsilon_{it} - \bar{\varepsilon}_i\right) \xrightarrow{D} W(z) - zW(1) \equiv B(z), \tag{3}$$

where $W(z)$ is a standard Brownian motion on the unit interval, with $W(z) \sim N(0,z)$, and $B(z)$ is a standard Brownian bridge, or tied Brownian motion. For all $z \in [0,1]$, $B(z)$ has an asymptotic Gaussian distribution, with mean zero and standard deviation $\sqrt{z(1-z)}$ (Bhattacharya and Waymire, 1990). For a given z – that is, to test for a break point in the model at a fixed and known date – an asymptotic 95% confidence interval for $B(z)$ is $\pm 1.96\sqrt{z(1-z)}$. To check for an unknown break point, a statistic based on the supremum norm,

$$Q_T = \sup_{z \in [0,1]} \left| B_T(z) \right| \tag{4}$$

has an asymptotic 5% critical value of 1.36 (Ploberger and Krämer, 1992).

We can use consistently estimated residuals and consistently estimated standard errors to obtain sample analogues to these asymptotic Brownian bridges. This gives

$$B_{iT}(z) \equiv \frac{1}{\sqrt{T}\hat{\sigma}_i} \sum_{t=1}^{[zT]} (\hat{\varepsilon}_{it} - \overline{\hat{\varepsilon}}_i) \xrightarrow{D} B(z), \tag{5}$$

also uniformly in $z \in [0,1]$, so long as the model specification is correct and the parameters are constant across time periods. This statistic is a single equation first-order specification/parameter stability statistic since it is based on the first-order moment conditions, $E(\varepsilon_{it}) = 0 \ \forall \ i, t$. A system-wide first-order specification/parameter stability statistic can be defined by

$$B_T(z) \equiv \frac{1}{\sqrt{T}} \sum_{t=1}^{[zT]} \left[\frac{1}{\sqrt{n_x}} \sum_{i=1}^{n_q} (\hat{\xi}_{it} - \overline{\hat{\xi}}) \right] \xrightarrow{D} B(z), \tag{6}$$

where $\hat{\boldsymbol{\xi}}_t = \hat{\boldsymbol{\Sigma}}^{-½} \hat{\boldsymbol{\varepsilon}}_t$ is the t^{th} estimated standardized error vector and $\overline{\hat{\xi}} \equiv \sum_{t=1}^{T} \sum_{i=1}^{n_q} \hat{\xi}_{it} \big/ n_x T$.

Similar methods apply to second-order stationarity and parameter staibility. We focus on system-wide statistics. Let $\boldsymbol{\Sigma}$ be factored into $\boldsymbol{L}\boldsymbol{L}^{\top}$, where $\boldsymbol{L}$ is lower triangular and nonsingular. Define the random vector $\boldsymbol{\xi}_t$ by $\boldsymbol{\varepsilon}_t = \boldsymbol{L}\boldsymbol{\xi}_t$. In addition to the assumptions above, add $\sup_{i,t} E(\varepsilon_{it}^4) < \infty$. Estimate the within-period average sum of squared standardized residuals by

$$\hat{\upsilon}_t = \tfrac{1}{n} \hat{\boldsymbol{\xi}}_t^{\top} \hat{\boldsymbol{\xi}}_t = \tfrac{1}{n} \hat{\boldsymbol{\varepsilon}}_t^{\top} \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\varepsilon}}_t , \tag{7}$$

where $\hat{\boldsymbol{\varepsilon}}_t$ is the vector of consistently estimated residuals in period *t* and $\hat{\boldsymbol{\Sigma}} = \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t^{\top} \big/ T$ is the associated consistently estimated error covariance matrix. The mean of the true υ_t is one for each *t*, and the martingale difference property of $\boldsymbol{\varepsilon}_t$ is inherited by $\upsilon_t - 1$. A consistent estimator of the asymptotic variance of υ_t is

$$\hat{\sigma}_{\upsilon}^2 = \frac{1}{T} \sum_{t=1}^{T} (\hat{\upsilon}_t^2 - 1). \tag{8}$$

A system wide second-order specification/parameter stability test statistic is obtained by calculating centered and standardized partial sums of $\hat{\upsilon}_t$,

$$B_T(z) = \frac{1}{\sqrt{T}\hat{\sigma}_{\upsilon}} \cdot \sum_{t=1}^{[zT]} (\hat{\upsilon}_t - 1) \ \xrightarrow[T \to \infty]{D} \ B(z), \tag{9}$$

uniformly in $z \in [0,1]$, where the limiting distribution on the far right follows from the identity $\overline{\hat{\upsilon}} \equiv \sum_{t=1}^{T} \hat{\upsilon}_t \big/ T \equiv 1$.