BACKGROUND PAPER FOR ARE SEMINAR, OCT 28 2011

RATIONAL EXAGGERATION IN INFORMATION AGGREGATION GAMES

GORDON C. RAUSSER, LEO K. SIMON, AND JINHUA ZHAO

SEPTEMBER 2010

ABSTRACT. This paper studies a class of information aggregation models which we call "aggregation games." It departs from the related literature in two main respects: information is aggregated by averaging rather than majority rule, and each player selects from a continuum of reports rather than making a binary choice. The game models in a stylized way the operation of a class of aggregating institutions that play an increasing role in modern economies. A leading example is the process by which LIBOR rates are determined. Each of a finite collection of players receives a private signal, then submits a report to the center, who then makes a decision based on the average of these reports. The essence of an aggregation game is that heterogeneous players engage in a "tug-of-war," as they attempt to manipulate the center's decision process by mis-reporting their private information. When players have distinct biases, almost all of them rationally exaggerate the extent of these biases. The paper focuses primarily on games with a small number of players. We identify a class of "anchored" games with quadratic payoffs for which sharp comparative statics results can be obtained. These results relate to the impacts of changing the number of players, the degree of player heterogeneity and the space of admissible announcements. We also show that as the number of players increases without bound, the relationship between players' signals and the outcome becomes more and more tenuous, precisely as the relationship between these signals and the true state becomes more and more clearcut.

KEYWORDS: information aggregation; majority rule;LIBOR;Baltic Dry Index; proportional representation; mean versus median mechanism; strategic communication; incomplete information games; strategic information transmission

JEL CLASSIFICATION: F71, D72, D82.

The authors are, respectively, Robert Gordon Sproul Distinguished Professor, Adjunct Professor and Associate Professor. Rausser and Simon are at the Department of Agricultural and Resource Economics, University of California at Berkeley. Zhao is at the Department of Economics and the Department of Agricultural, Food and Resource Economics, Michigan State University.

The authors are grateful to seminar participants at the University of Washington, Iowa State University, the University of Cassino (Italy) and the University of California at Berkeley. Qu Tang provided invaluable research assistance. The authors have benefited from conversations with many colleagues, especially Susan Athey, Carlo Russo, Rachael Goodhue, George Judge, Larry Karp, Wolfgang Pesendorfer, Chris Shannon and Sofia Villas-Boas.

1. INTRODUCTION

We model a class of games that are naturally described as *aggregation games*. There is a finite collection of players. Each player is characterized by two parameters: the first is a privately observed signal, identified with the player's *type*; the second is an observable characteristic, such as a voting record, profession, income or location. Players' types are continuously distributed on a compact interval; the distribution of types is common knowledge. Players simultaneously observe their signals, then submit reports to a central authority, who makes a decision which affects all of them. Reports are rejected unless they lie in a prespecified compact interval. The authority's decision rule is fixed and commonly known. The defining property of an aggregation game is that two of its key components—the center's decision and players' utilities—depend on players' realized types only through the mean of these realizations. More formally, a player's strategy in an aggregation game is to make a report based on his type. The center maps the mean of these reports, paired with the vector of observable characteristics, to some interval. Each player's utility depends on his own observable characteristic, the center's decision and the mean of players' privately observed signals.

Our model has at least two alternative interpretations. The first is Bayesian: the center treats players' type reports as a sample of signals drawn from a distribution whose unknown mean is payoff relevant. Under this interpretation, the distribution of player types is the marginal joint distribution of the sample data. The center's decision rule depends on the mean of players' announcements, which it treats as an estimate of the unknown population mean. Each player's utility depends on the center's choice, as well as the (unobservable) mean of all players' signals, which is a sufficient statistic for the population mean of the signal distribution. The third argument of a player's utility is his own observable characteristic, which is the individual's subjective bias relative to the best available estimate of the truth.

In our second, non-statistical interpretation, the center aggregates information but does not draw inferences from it. Again, each player's type is the realization of a random variable, but in this case each realization is interpreted as the true value of a single component of some vector. As before, the center's decision is based on the mean value of players' reported types, which is in this case interpreted as a summary value of a composite assessment. The utility that each player associates to the vector depends on this summary value, but is also subject to idiosyncratic bias.

This paper contributes to an extensive literature on information aggregation that goes back to Condorcet (1785). A common theme of this literature, reviewed in Sec. §2, is that individuals send messages to the center, which are somehow aggregated and mapped to an outcome that affects everybody. The question is then asked: how well does the aggregation process work? Specifically, under what circumstances does the resulting outcome coincide with the one that would have been selected by a aggregate-welfare-maximizing decision maker with full access to the private information on which agents base their messages? The institution/aggregation mechanism which has been examined most thoroughly is majority rule, especially in the context of elections. We examine an alternative mechanism—report averaging. While averaging is arguably more significant in practice than majority rule as a tool for making decisions, the former has received much less attention from researchers than the latter.

Increasingly many institutions in modern economies use report averaging to aggregate information provided by industry or market participants. This paper studies the incentives facing an agent participating in such an institution to misreport her information, when she has an interest in the outcome of the aggregation process. For example, if an agent has a bias in favor of outcomes that exceed the one that maximizes aggregate welfare given agents' aggregate private information, she then has an incentive to upwardly bias her reports; that is, she can be expected to "rationally exaggerate" her information. Indeed, there have been widespread concerns about attempts to manipulate the outcomes of a variety of report-averaging institutions. For economists schooled in mechanism design, a natural response to these concerns is to focus on the design of incentive schemes that would reverse engineer via the revelation principle the mis-reporting process. In practice, however, mechanistic report-averaging has proved extremely resilient as an aggregating instrument, in spite of its obvious deficiencies. Indeed, the aggregation institutions we discuss below are necessarily non-strategic, because the economic actors that utilize their products require them to be so. In many industries, a single aggregator becomes the scorekeeper upon which all users rely. The groundrules upon which these aggregators' existence is premised require them to provide their patrons and/or their clients with mechanical, transparent service which is passive rather than strategic. Accordingly, real-world information aggregators have responded to concerns about manipulation of their processes either by eliminating outlier reports or by attempting to enforce compliance in various ways, rather than to adopt more sophisticated aggregation tools in which the center acts strategically. Since report averaging is clearly a robust institution in modern economies, our focus in this paper is on the implications of exaggeration when the center acts non-strategically.

1.1. **Empirical examples of information aggregation by averaging.** Judged by its financial impact, by far the most significant example of information aggregation by averaging is the process by which LIBOR is determined.[1] MacKenzie (2008) estimates the value of financial contracts whose interest rates are based on Libor to be approximately $300 trillion. Every weekday, eight to sixteen leading banks submit estimates to Thomson Reuters of the interest rates at which they could borrow money from other banks for various durations.[2] Reuters discards the lowest and highest quartiles of the estimates it receives, and declares the average of the remaining estimates to be the daily Libor rate for that currency/duration. This method of determining Libor rates has been widely criticized as manipulable. In an influential WSJ article, Mollenkamp and Whitehouse (2008) argued that during the recent financial crisis, several banks were reporting borrowing costs that were significantly lower than their true costs, in order to appear more financially sound than they in fact were. Snider and Youle (2010) provides persuasive documentation of exaggeration, presenting "suggestive evidence that misreporting incentives are partially driven by member bank portfolio positions" (p. 3). In spite of extensive calls for reform, the institutional underpinnings of Libor remain basically unchanged.

Our model can be viewed as a stylized representation of the LIBOR rate determination process, although simplified in one important respect. The players in our model are the contributing banks; the center is Thomson Reuters; the player's *type* is a summary statistic for the bank's private information about market conditions relating to interest rate determination; the player's observable characteristic is a parameter indicating whether the bank has a bias relative to the industry in favor of a higher or lower interest rate for a particular duration. Since this indicator of bias will typically depend on public information such as balance sheet considerations and portfolio positions, it is natural to model it as commonly known. Our model simplifies the Libor process by assuming

[1]"Judged by the amount of money directly dependent on it, the British Bankers' Association's London Interbank Offered Rate (LIBOR) matters more than any other set of numbers in the world" (MacKenzie, 2008).

[2]Banks' reports are responses to the following question: "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?" (British Banking Assoc, n.d.)

that a contributor's report will be accepted or rejected with certainty, depending on whether or not it belongs to an exogenously specified interval; by contrast, the (*ex ante*) probability that a particular bank's actual Libor quote will be accepted is endogenously determined, depending on the location of the *ex ante* unknown boundaries of the interquartile range. This distinction will be less significant, the greater is the extent to which banks can predict the location of these boundaries. In fact, it appears that they can do so with some accuracy. Snider and Youle (2010) find strong evidence that Libor quotes are "bunched" at the ex-post boundaries of the inter-quartile range: moreover, they present evidence that banks with financial incentives to raise (lower) Libor rates submit quotes near the upper (lower) boundaries of this range.

Many other important indices are computed by averaging the reports provided by interested parties to a central "scorekeeper." The Baltic Dry Index (BDI) is considered to be one of the purest leading indicators of economic activity (Gross, n.d.).[3] It is determined by aggregating the responses by shipping brokers to daily questions about how much it would cost to book various cargoes of raw materials on various routes. Like Libor, the BDI anchors many financial contracts: for example, bulk shippers and carriers regularly trade "freight forward contracts" linked to the BDI, to hedge against movements in spot freight rates (Leach, 2010). Since the index is maintained by and for professionals in the shipping business, brokers clearly have incentives to manipulate it by exaggerating their daily responses. In fact, however, the index is generally regarded as extremely reliable (Hansen, n.d.).

By contrast, the natural gas price index computed by Platts, a source of benchmark price assessments for physical energy markets, was famously distorted between 2001 and 2005 as a result of exaggerated reports. The Commodity Futures Trading Commision (CFTC) levied fines totaling $350 million in actions against energy suppliers alleging attempted manipulation of the price of natural gas. Most of these cases focused on attempted manipulation by falsely reporting natural gas trading information to energy index firms such as Platts. The affected Platts reports sent false signals to other market participants that supplies were significantly tighter than expected, and prices rose dramatically as a consequence (USGAO, 2007; Jickling, 2008). Several Enron executives were jailed as a result of the CFTC's investigations, and Platts was obliged to redesign the

[3]"It represents the cost paid by an end customer to have a shipping company transport raw materials across seas on the Baltic Exchange, the global marketplace for brokering shipping contracts" (Wikinvest, n.d.).

data acquisition procedure on which its natural gas price index was based, to rely less on industry reports, and more on verifiable data.

Under the Agricultural Marketing Act of 1946, the USDA's Agricultural Marketing Service (AMS) has been collecting livestock and meat price and related market information on a voluntary basis. Again, there is evidence of rational exaggeration by information providers. Koontz (1999), comparing voluntary AMS price reports against transaction prices from objective sources, found evidence that voluntary reporting was inefficient during times when prices were changing appreciably. In particular, the fed cattle price range reported by USDA did not increase fast enough with rising prices, nor decline fast enough with declining prices. He concluded that this could be a result of selective price reporting by both meat packers and feedlots when markets were moving against them.

In all of the examples discussed above, the number of agents that contribute reports is relatively small. In other instances, the number of report contributors is much larger. For example, the average of students' evaluations of their professors play an increasingly important role in academics' tenure and promotion decisions. Contingent valuation studies aggregate the opinions of multiple responders in order to assign values to non-market resources such as environmental goods, and to assess the damage due to contamination, oil spills, etc (Carson, Flores and Meade, 2001). Increasingly, consumers rely on summary indicators provided by online services such as Yelp, Trip Advisor, Rate My Professor, etc., which aggregate reviews contributed by multiple patrons of movies, restaurants, hotels and a host of other goods and services.

1.2. **Structure of the paper.** The paper is organized as follows. A † sign after the title of a proposition indicates that its proof is in the appendix. When propositions follow immediately from arguments in the text, formal proofs are omitted. For concreteness, we will sometimes refer to the players in our game as "right-wingers" and "left-wingers," and distinguish between moderates and extremists. Right-wingers want to distort to the right the average signal that the center receives, and extremists want to distort more than moderates.

Sec. §2 relates our model to the literature. In §3 we introduce our model in its most general form and prove that every aggregation game has a pure strategy equilibrium in which players' strategies are monotone in their types. This result highlights the pivotal role in our model played by the bounds imposed on acceptable reports. In the absence of such bounds, unless all players have *ex ante* identical characteristics, right- and left-wingers would engage in an endlessly escalating tug-of-war: the former would distort their signals further and further to the right, in order to offset increasingly magnified leftward distortions by the latter. A central result of our paper is that when players are heterogeneous, all but at most one must be constrained with positive probability by one of the boundaries, in order to break this diverging cycle. Thus, some degree of information loss is a necessary condition for equilibrium.[4] Sec. §4 demonstrates that incentives to mis-report do not arise when players have *ex ante* identical characteristics. In §5-§7, we focus on small "quadratic" games. The ultimate goal in these sections is to explore how the information losses due to boundary constraints depend on fundamental parameters. In order to obtain determinate comparative statics results, we impose further restrictions: we assume that players' utilities are "biased quadratic loss functions."[5] In §5, we develop machinery that will be applied in the comparative statics analysis in §6 and §7. Every quadratic game has a unique pure strategy equilibrium, in which a player's *unconstrained* strategy is an affine function of his type.

Quadratic games are particularly tractable when there is one player whose affine strategy is never constrained by the announcement bounds. We call this player the "anchor" and identify a class of games called anchored games. In §6, we study n-player anchored games that are symmetric in a strong sense: there is a right-wing faction and a precisely symmetric left-wing faction. Several of the properties of these games are quite striking. Outcomes, payoffs and aggregate welfare are all independent of the bounds on the announcement space, provided these bounds contain the type space and preserve symmetry. To explore in a controlled environment the effect of increasing n, we clone repeatedly a small set of players until the point at which some players are constrained with probability one, thus generating a finite sequence of increasingly large games. If the type distribution is uniform, players' payoffs initially decline due to increased information losses; eventually, however, this decline is reversed as the law of large numbers asserts itself and players' distortions tend more and more to offset each other. We also investigate the impact of player heterogeneity:

---

[4] The role of a compact message space in limiting information transmission has been noted in the literature, in contexts that differ from ours. See for example, Ottaviani and Squintani (2006).

[5] We use the term "biased quadratic loss function" to denote a loss function $L(x,\bar{x},b) = -((\bar{x}+b)-x)^2$. in which the target value is the truth $\bar{x}$ plus a bias b. This specification is standard in the costless information transmission literature. See for example Crawford and Sobel (1982) and Morgan and Stocken (2008), and the references cited in their fn. 10.

intuitively, payoffs decline as heterogeneity increases. However, if initially the two factions are sufficiently polarized, payoffs will actually increase when we increase the heterogeneity of each *faction*, holding constant the faction means. §7 studies a quite different class of anchored games, in which the upper bound on the announcement space is so high that it never binds in equilibrium. Games in this class are anchored by the player with the highest observable characteristic. In spite of the obvious structural differences, this class of games has properties that are remarkably similar to those of symmetric games. In §8, we examine the properties of our model when the number of players increases without bound. (The games examined in §8 are much larger than the largest ones considered in §6.) A recurring theme in the information aggregation literature is that political institutions such as elections effectively aggregate private information when the number of participants is very large. Based on this literature, one would expect that since players' signals are i.i.d. around an unknown population mean, our model with a large number of players would implement with probability approaching one an outcome very close to this mean. In fact, however, the outcomes in our model converges to a constant which is weighted average of the lower and upper bounds on admissible reports; the weight depends only on the proportion of right-wingers to left-wingers, and is independent of the population mean. Thus, the relationship between signals and outcomes becomes more and more tenuous as the relationship between signals and the true state becomes more and more clearcut. In the limit, the impact of players' aggregated private information on outcomes is entirely obliterated. Sec. §9 concludes.

2. Related literature

In assessing the prior literature, it is helpful to classify it along three dimensions. The first distinguishes between models of majority rule versus averaging mechanisms; the second between models in which players' preferences prior to receiving their private signals are homogeneous or heterogeneous; the third between choice sets containing either two or a continuum of options. We discuss a small selection of papers that relate most closely to our analysis.[6]

The related literature focuses primarily on the informational efficiency of voting under majority rule. The classical Condorcet Jury Theorem established conditions under which, when voters with identical preferences select non-strategically (or *sincerely*) between two alternatives based on

[6]Piketty (1999), Gerling et al. (2005) and Dewan and Shepsle (2008) all survey the literature quite extensively.

their private information, and the majority prevails, then as the number of voters increases without bound, information is in the limit perfectly aggregated, in the sense that the majority's choice coincides with the choice that would be taken if all private information were publicly available. (Feddersen and Pesendorfer (1997) [FP] later call this property "full information equivalence.") Austen-Smith and Banks (1996) [AB] study the relationship between sincerity and rationality. Under majority rule, rationality dictates that one should decide how to vote conditional on the presumption that one's vote is decisive (or *pivotal*). Conditional on being pivotal, one can make inferences about the distribution of other players' realized signals and thus about the true state of the world. Rationality requires that these inferences be taken into account when deciding how to vote. AB then show that for three very simple specifications, voting sincerely is, except in very special circumstances, incompatible with voting *informatively*, i.e, in a way that depends nontrivially on one's private signal. While AB focused on small games, FP explores the implications of pivotality in large ones. FP's specification of players' preferences is quite similar to ours, except that their center chooses between two alternatives according to majority rule.[7] FP consider a sequence of games in which n increases without bound; when players condition on pivotality, their limit game exhibits full information equivalence. This property is quite robust. For example, McLennan (1998) considers sequences of games with increasing n in which players have common preferences; full information equivalence again holds in the limit under very general conditions. Lohmann (1993) identifies conditions under which the same property holds when players demonstrate rather than vote.

As we noted in §1, matters are quite different when the center averages players' reports rather than applies majority rule. A major source of the difference is that pivotality no longer plays any role, since the leverage that an individual has on the center's decision is now independent of the actions taken by other players. Consequently, players simply condition their actions on their private signals, just as they do under Condorcet's sincere voting. One of very few papers that focuses exclusively on the averaging mechanism is Morgan and Stocken (2008) [MS]. MS's constituents, who have varying degrees of bias, are polled about the state of the world. Each one receives a binary signal about this state, and sends one of two possible reports. The center aggregates these reports and chooses a policy accordingly. A right-winger who receives a left-favoring signal is

[7] A second difference is that our players' biases are publicly known while theirs are private information.

tempted to mis-report in order to bias the center's decision to the right. If n is small enough, he will be deterred from doing so by the possibility that he might over-shoot, shifting the policy to the right of his preferred location. As n increases, the possibility of overshooting diminishes along with each individual's leverage over the ultimate policy decision, so that more and more constituents vote according to their biases rather than their information.

MS demonstrate that even when n is large, full information equivalence can be restored through stratified sampling: by eliminating the responses of those identifiable as strongly biased based on observable criteria, the center in effect limits the size of the game, restoring the remaining centrists' leverage over the outcome, which induces them to respond based on their realized information rather than their biases, in order to avoid overshooting. MS and our paper are similar in many respects. In particular, both highlight the negative impact on information transmission of the averaging mechanism. The primary difference between MS and our paper is that their players make a binary choice while our players receive signals and select responses from a continuum of options. Overshooting is not a deterrent in our model; our players can mis-report to whatever extent they desire, except when they are constrained by the announcement bounds. More important, the notion of *rational exaggeration*, which is central to our paper, has no meaning when agents make binary choices.

Gruner and Kiel (2004) [GK] compare the performance of games in which the center chooses either the median or the mean of players' reported private information. Their median model corresponds to majority rule; their mean model corresponds to our averaging mechanism. In contrast to the papers discussed above, GK's players choose from a continuum of reports rather than make a binary choice. In contrast to our model, the biases of GK's players are proportional to their private signals; with this non-standard assumption, GK can obtain existence without requiring the announcement space to be compact. GK's formal results focus exclusively on the relationship between the magnitude of players' biases and the relative performance of the two mechanisms. Their major conclusion is that the mean mechanism outperforms the median iff agents' biases are sufficiently small. Indeed, as in our paper, the mean mechanism achieves the first best when all biases are zero. While they do not study formally the comparative statics effects of n, GK do provide examples showing that with biased players, the performance of the mean mechanism deteriorates as n increases from 3 to 7.

GK's examples illustrate nicely some of the themes that are central to this paper. The mean dominates the median when players have common interests because the former utilizes all reported information and agents have no incentive to misreport; by contrast, the median mechanism utilizes only the reported information that the median player provides, so that perfectly good information is ignored. When players have significant biases, however, this strength of the mean mechanism is also its weakness, which is exacerbated as n increases. As noted, an individual's leverage over the center's decision declines with n, requiring more and more exaggeration in order to accomplish a given shift; in addition, under the mean mechanism, there is the "tug-of-war" aspect of exaggeration that we discuss above on pp. 5-6.[8] Both effects diminish the accuracy of reported information. Under the median mechanism, on the other hand, the median player has one-to-one leverage: she does not have to engage in a tug-of-war with other players; nor is her leverage diluted by n. Since players under this mechanism condition their reports on being pivotal (i.e., on being the median player), the information they report is much closer to the truth.

Still another framework is presented by Razin (2003), in which an electorate with common preferences chooses between two candidates. Each voter receives a private signal that is correlated with the ideal policy location. The winning candidate treats the magnitude of his victory as a guide for setting policy. Because both candidates have ideological biases, while the population is ideologically neutral, the policy that would be selected if all private information were revealed would be extreme relative to the electorate's common bliss point. Depending on the degree to which candidates are polarized, and the responsiveness of their policy choices to election results, there will be a conflict between voters' motivation to select the more appropriate candidate, conditional on being pivotal, and their unconditional motivation to correct for the winning candidate's ideological bias. From our perspective, the primary interest of Razin's paper is that it melds into one mechanism the averaging and majority rule mechanisms that we seek to compare.

3. The Model

An aggregation game is an incomplete information simultaneous-move game among n players, indexed by $r = 1,\dots,n$. For any $\mathbf{x} \in \mathbb{R}^n$ the symbol $\mu(\mathbf{x})$ will denote the average of $\mathbf{x}$'s components.

[8] Ortuno-Ortin (1997) examines the incentives to exaggerate in a model of elections with proportional representation.

Player characteristics: We assume that each player is characterized by an *observable characteristic* and a *type*. Player r's *type* is $\theta_r \in \mathbb{R}$, which is his private information. We assume that the θ_r's are identically, independently and continuously distributed on the compact interval $\Theta \equiv [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$, with $\overline{\theta} > \underline{\theta}$. Let $h(\cdot)$ denote the density, and $H(\cdot)$ the c.d.f., of players' types. Let $\boldsymbol{\Theta} = \Theta^n$ denote the space of *type profiles*, with generic element $\boldsymbol{\theta}$. Similarly, let $\boldsymbol{\Theta}_{-r} = \Theta^{n-1}$ be the space of types for players other than r, with generic element $\boldsymbol{\theta}_{-r}$. For $\boldsymbol{\theta}_{-r} \in \boldsymbol{\Theta}_{-r}$, let $\mathbf{h}_{-r}(\boldsymbol{\theta}_{-r}) = \prod_{i \neq r} h(\theta_i)$. When we integrate w.r.t. either player r's type or all other players' types, we will use, respectively, the variants ϑ_r and $\boldsymbol{\vartheta}_{-r}$ of θ_r and $\boldsymbol{\theta}_{-r}$ to distinguish dummy variables of integration.

Player r's observable characteristic is denoted by $k_r \in \mathbb{R}$ and is interpreted as r's bias w.r.t. revealed information: a player whose characteristic is positive prefers the center to over-estimate the mean of players' types. We refer to the vector $\mathbf{k} = (k_r)_{r=1}^n$ as the *observable characteristic profile*. To avoid special cases and/or additional notation:, we impose two restrictions on observable characteristics: players' biases cancel each other out in the aggregate and they are distinct.

Assumption A1: (i) $\sum_i k_i = 0$; (ii) $i \neq r \implies k_i \neq k_r$.

Restriction (i) yields a clean expression for welfare while (ii) ensures uniqueness. Part (ii) will be relaxed in §4 as well as §6.1 and §7.1.

The utility function: The *utility function* is a mapping $u : T \times \boldsymbol{\Theta} \times \mathbb{R} \to \mathbb{R}$, where $T \subset \mathbb{R}$ is compact. The scalar first argument of u can be interpreted as the decision taken by a central authority, in response to information provided by the players: $u(\tau, \boldsymbol{\theta}, k)$ is the utility to a player with observable characteristic k, when the central authority's decision is τ and the vector of unobservable characteristics is $\boldsymbol{\theta}$. The essence of an aggregation game is that a player's type affects his utility only through its effect on the average of all players' types. Specifically, we impose

Assumption A2: $\mu(\boldsymbol{\theta}) = \mu(\boldsymbol{\theta}') \implies u(\tau, \boldsymbol{\theta}, k) = u(\tau, \boldsymbol{\theta}', k)$

In the formal development below, we will, depending on which is more convenient, write the second argument of u either as the vector $\boldsymbol{\theta}$ or the scalar $\mu(\boldsymbol{\theta})$.

Pure strategies: Reports are rejected by the center unless they belong to a prespecified compact interval, denoted by $A = [\underline{a}, \overline{a}]$. Given the structure of our model, a player whose unconstrained optimal report exceeds $\bar{a}$ necessarily weakly prefers to have a report of $\bar{a}$ accepted than to have his report rejected. Accordingly, to streamline the exposition, we impose as a restriction that each player must choose a report in A. Formally, we define a *pure strategy* for player r to be a function $s_r : \Theta \to A$, where $s_r(\theta_r)$ denotes the announcement of player r when his type is θ_r. (Henceforth, the symbol s_r will denote a *function* from types to A, while a_r will denote a particular value of $s_r(\theta_r)$.) The vector $\mathbf{s} = (s_1, \ldots, s_n)$, called a *pure strategy profile*, is thus a mapping from $\boldsymbol{\Theta}$ to $\mathbf{A} = A^n$. A pure strategy $s_r(\cdot)$ is said to be *monotone* if it is nondecreasing and strictly increasing except when $s_r(\cdot)$ is at the boundary of A. Since the space A is bounded both above and below, if s_r is monotone, there exists a *low threshold type* $\utilde{\theta}_r \in [\underline{\theta}, \overline{\theta}]$ and a *high threshold type* $\tilde{\theta}_r \in [\underline{\theta}, \overline{\theta}]$ such that s_r equals $\underline{a}$ on $[\underline{\theta}, \utilde{\theta}_r)$, is strictly increasing on $(\utilde{\theta}_r, \tilde{\theta}_r)$ and equals $\bar{a}$ on $(\tilde{\theta}_r, \overline{\theta}]$.[9] Formally,

$$\utilde{\theta}_r(s_r) = \begin{cases} \underline{\theta} & \text{if } s_r(\underline{\theta}) > \underline{a} \\ \sup \{\theta \in \Theta : s_r(\theta) = \underline{a}\} & \text{if } s_r(\underline{\theta}) = \underline{a} \end{cases}, \tag{1a}$$

$$\tilde{\theta}_r(s_r) = \begin{cases} \overline{\theta} & \text{if } s_r(\overline{\theta}) < \bar{a} \\ \inf \{\theta \in \Theta : s_r(\theta) = \bar{a}\} & \text{if } s_r(\overline{\theta}) = \underline{a} \end{cases}. \tag{1b}$$

<u>The outcome function</u>: The *outcome function*, $t : A^n \times \mathbb{R}^n \to \mathbb{R}_+$, maps player announcements and the vector of observable characteristics to actions by the central authority. Our center aggregates information mechanically rather than strategically. Indeed, we restrict outcome functions to be *complete information socially efficient (CISE)*, meaning that if players were to truthfully reveal their types on average, the outcome t would maximize social welfare, defined as the average of players' individual utilities. That is, defining the *social welfare function* as

$$w(\tau, \boldsymbol{\theta}, \mathbf{k}) = \sum_i u(\tau, \boldsymbol{\theta}, k_i)/n, \tag{2}$$

the CISE outcome function is $t(\boldsymbol{\theta}, \mathbf{k}) = \text{argmax } w(\cdot, \boldsymbol{\theta}, \mathbf{k})$. We refer to an outcome implemented by a CISE outcome function as a *CISE outcome*. It follows from assumption A2 that CISE outcomes

[9] Either one of the half-open intervals can be empty. For example, if $s_r(\cdot) > \underline{a}$ on Θ then the interval $[\underline{\theta}, \utilde{\theta}_r(s_r))$ is empty.

depend on players' announcements only through their average, i.e.,

$$\mu(\mathbf{a}) = \mu(\mathbf{a}') \quad \Longrightarrow \quad t(\mathbf{a},\mathbf{k}) = t(\mathbf{a}',\mathbf{k}) \tag{3}$$

Once again, we will write the first argument of t as either an n-vector or its average, depending on convenience. Also, since $\mathbf{k}$ is typically fixed, we will often omit t's second argument.

Player's expected payoff functions: Player r's expected payoff function, U_r, maps his own announcement and type into his utility, given other players' strategies. Our expression for U_r suppresses r's observable characteristic and the outcome function. Formally, given a profile, $\mathbf{s}_{-r}$, of strategies for players other than r, player r's expected payoff function $U_r : A \times \Theta \to \mathbb{R}_+$ is

$$U_r(a,\theta;\mathbf{s}_{-r}) \quad = \quad \int_{\mathbf{\Theta}_{-r}} u\left(t\left((a,\mathbf{s}_{-r}(\boldsymbol{\vartheta}_{-r})),\mathbf{k}\right),(\theta,\boldsymbol{\vartheta}_{-r}),k_r\right) d\mathbf{h}_{-r}(\boldsymbol{\vartheta}_{-r}). \tag{4}$$

In what follows, the derivative $\frac{\partial U_r}{\partial a}$ will play an important role; when confusion can be avoided, we will abbreviate this expression to U_r'.

Equilibrium: A *monotone pure strategy Nash equilibrium* (MPE) for an aggregation game is a monotone strategy profile $\mathbf{s}$ such that for all r, $\theta \in \Theta$, and $a \in A$, $U_r(s_r(\theta),\theta;\mathbf{s}_{-r}) \geq U_r(a,\theta;\mathbf{s}_{-r})$.

We make the following additional assumptions throughout the paper.

Assumption A3: The density, $h(\cdot)$, of players' types is bounded.

Assumption A4: The utility function u is bounded and thrice continuously differentiable. For each k and $\mu(\boldsymbol{\theta})$, $u(\cdot,\mu(\boldsymbol{\theta}),k)$ is strictly concave.

Assumption A5: For all $(\tau,\mu(\boldsymbol{\theta}),k)$, (i) $\frac{\partial^2 u(\tau,\mu(\boldsymbol{\theta}),k)}{\partial\tau\partial\mu(\boldsymbol{\theta})} > 0$, and (ii) $\frac{\partial^2 u(\tau,\mu(\boldsymbol{\theta}),k)}{\partial\tau\partial k} > 0$.

Assumption A6: For all k and $\boldsymbol{\theta}$, $u(t(\cdot,\mathbf{k}),\mu(\boldsymbol{\theta}),k)$ is strictly concave in $\mu(\mathbf{a})$, the average of players' announcements.

Some additional assumptions will be introduced later. Whenever a list of assumptions is not explicitly included in the statement of a proposition below, this means that A1-A6 are satisfied.

Assumptions A4 and A5(i), together with the fact that $t(\cdot)$ is CISE, imply that:

$$t(\cdot,\mathbf{k}) \text{ is strictly increasing and continuously differentiable in } \mu(\mathbf{a}). \tag{5}$$

Assumption A6 implies that

$$U_r \text{ is strictly concave w.r.t. its first and third arguments.} \tag{6}$$

Assumption A3 is required to ensure that pure-strategy equilibria exist. Assumption A5 states that players with higher unobservable and/or observable characteristics derive higher marginal utility from an increase in the central authority's decision.[10] Assumption A6 is not entirely straightforward. It states that $\frac{\partial^2 u}{\partial(\mu(\mathbf{a}))^2} = \frac{\partial^2 u}{\partial t^2}\left(\frac{\partial t}{\partial \mu(\mathbf{a})}\right)^2 + \frac{\partial u}{\partial t}\frac{\partial^2 t}{\partial(\mu(\mathbf{a}))^2}$ is globally negative. However, since u is not monotone in t, the second term cannot be signed in general.[11] We make this assumption to simplify the analysis. In particular, since $U_r' = \int_{\Theta_{-r}} \frac{\partial u}{\partial t}\frac{dt}{da} d\mathbf{h}_{-r}(\boldsymbol{\vartheta}_{-r})$, assumption A6 implies that for all r, all θ and all $\mathbf{s}_{-r}$, $U_r(\cdot,\theta;\mathbf{s}_{-r})$ is strictly concave in a. Thus, each player has a unique optimal response to other players' strategies.

From (5), t is strictly increasing; it follows, therefore, from (4) and A5(i) that

$$\text{for all } r, \text{ all } a, \text{ all } \theta \text{ and all all } \mathbf{s}_{-r}, \quad \frac{\partial^2 U_r(a,\theta;\mathbf{s}_{-r})}{\partial a \partial \theta} \quad > \quad 0. \tag{7}$$

Inequality (7) states that U_r satisfies Milgrom-Shannon's condition SCP-IR in $(a;\theta)$ (see fn. 10). In our context, this property implies Athey's sufficiency condition, SCC, for existence of a pure-strategy equilibrium, i.e., "the single crossing condition for games of incomplete information" (Athey, 2001, Definition 3). Athey's condition requires that U_r satisfies SCP-IR only if other players play non-increasing strategies. Our U_r's satisfy SCP-IR regardless of other players' choices.

Proposition 1 (Existence of an MPE):[†] *Every aggregation game has a monotone pure-strategy Nash equilibrium,* $\mathbf{s}$*, with the property that for each* r*,* s_r *is continuously differentiable on* $(\underset{\sim}{\theta}_r(\mathbf{s}), \tilde{\theta}_r(\mathbf{s}))$.

The Kuhn-Tucker conditions defining r's optimal strategy s_r are, for all $\theta \in \Theta$,

$$s_r(\theta) \quad = \quad \begin{cases} a & \text{if } U_r'(a,\theta;\mathbf{s}_{-r}) = 0 \text{ and } a \in [\underset{\sim}{a}, \bar{a}] \\ \bar{a} & \text{if } U_r'(\bar{a},\theta;\mathbf{s}_{-r}) > 0 \\ \underset{\sim}{a} & \text{if } U_r'(\underset{\sim}{a},\theta;\mathbf{s}_{-r}) < 0 \end{cases} \tag{8}$$

[10] Assumption A5(i) is a strict version of the "single crossing property of incremental returns (SCP-IR)" (Milgrom and Shannon, 1994) in $(\tau;\theta)$ when the utility function is differentiable (Athey, 2001, Definition 1).

[11] A sufficient condition to ensure Assumption A6 will hold is that $t(\cdot,\mathbf{k})$ is linear.

The essence of an aggregation game is that heterogeneous players are engaged in a "tug-of-war," trying to influence the equilibrium outcome through their announcements. As soon as one player who prefers a higher outcome attempts to influence the center by increasing his announcement, another player who prefers a lower outcome will counter by decreasing hers. In the absence of bounds on announcements, this tug-of-war would go on endlessly. Thus, a necessary condition for existence of MPE is that the announcement space A be compact. The bounds on announcement space essentially limit how far players can go in mis-reporting their types. We will observe below that players with different observable characteristics are restricted by the bounds to different degrees, and certain player-types "do particularly well" in equilibrium. To clarify concepts, we introduce some definitions. We will say that player r's strategy $s_r(\cdot)$ is

(1) *nondegenerate (resp. degenerate)* if the interval $(\utilde{\theta}_r(s_r), \tilde{\theta}_r(s_r))$ is non-empty (resp. empty).

(2) is *constrained at* θ if the announcement $s_r(\theta)$ equals either $\underline{a}$ or $\overline{a}$,

(3) is *up-constrained* if $\utilde{\theta}_r(s_r) = \underline{\theta}$ and $\tilde{\theta}_r(s_r) < \overline{\theta}$,

(4) is *down-constrained* if $\utilde{\theta}_r(s_r) > \underline{\theta}$ and $\tilde{\theta}_r(s_r) = \overline{\theta}$,

(5) is *single-constrained* if it is either up-constrained or down-constrained,

(6) is *bi-constrained* if $\utilde{\theta}_r(s_r) > \underline{\theta}$ and $\tilde{\theta}_r(s_r) < \overline{\theta}$.

(7) is *almost-never-constrained* if $\utilde{\theta}_r(s_r) = \underline{\theta}$ and $\tilde{\theta}_r(s_r) = \overline{\theta}$,

Degenerate (resp. *almost-never-constrained*) strategies pick boundary (resp. interior) points of A with probability 1.[12] An MPE in which each player's strategy is non-degenerate is called an NMPE.

Prop. 1 established that players' equilibrium strategies are monotonic in types. We next establish that strategies are also monotone with respect to players' observable characteristics. That is, if $k_i > k_j$ but both players are of the same type, i's announcement will strictly exceed j's, except when both announcements are at the same boundary $\underline{a}$ or $\overline{a}$. Moreover, as n increases, the gap between i's and j's equilibrium announcements increases until one or both players' strategies become degenerate: if j's (resp. i's) first order condition is satisfied with equality for some type and n is large enough, i (resp. j) will announce the upper bound $\overline{a}$ (resp. lower bound $\underline{a}$) with probability one.

Proposition 2 (Monotonicity w.r.t. observable characteristics):[†] *If* $\mathbf{s}$ *be an MPE, then for all* $\varepsilon > 0$ *and for all* i *and* j *such that* $k_i - k_j > \varepsilon$,

[12]The distinctions made here relate to the concept of *informative* voting, which recurs throughout the information transmission literature. (It appears to have been introduced in Austen-Smith and Banks (1996).) Almost-never-constrained strategies are informative, and degenerate ones are uninformative; the remaining types are somewhere in between.

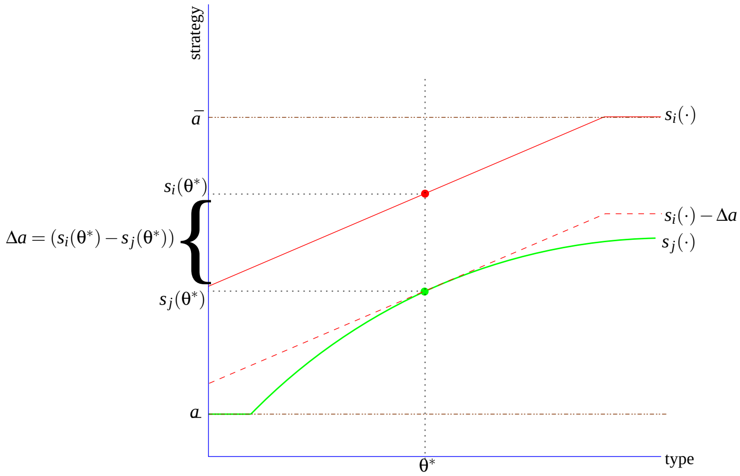

FIGURE 1. Intuition for Prop. 2

i) $\underset{\sim}{\theta}_i(\mathbf{s}) \leq \underset{\sim}{\theta}_j(\mathbf{s})$ *and* $\tilde{\theta}_i(\mathbf{s}) \leq \tilde{\theta}_j(\mathbf{s})$.

ii) $s_i(\cdot) > s_j(\cdot)$ *on the interval* $\left(\underset{\sim}{\theta}_j(\mathbf{s}), \tilde{\theta}_i(\mathbf{s})\right)$.

Further, there exists $N \in \mathbb{N}$ *such that*

iii) *if* $n > N$ *and* s_j *is non-degenerate, then* $s_i(\cdot) = \bar{a}$.

iv) *if* $n > N$ *and* s_i *is non-degenerate, then* $s_j(\cdot) = \underset{\sim}{a}$.

In the discussion of Prop. 2 that follows, we will say that the $\bar{a}$ (resp. $\underset{\sim}{a}$) constraint is *binding* on r at θ if the unconstrained optimal response of player r of type θ to $\mathbf{s}_{-r}$ strictly exceeds $\bar{a}$ (resp. is strictly less than $\underset{\sim}{a}$). Note significantly that by continuity, the $\bar{a}$ (resp. $\underset{\sim}{a}$) constraint is *not* binding on r at $\tilde{\theta}_r(\mathbf{s})$ (resp. $\underset{\sim}{\theta}_r(\mathbf{s})$). The key to the proof of Prop. 2 is the observation that if s_i and s_j form part of an equilibrium profile, then at any type θ^* belonging to the (necessarily nonempty) set $\Theta^* \equiv \arg\min\left(s_i(\cdot) - s_j(\cdot)\right)$,

$$\textit{either} \text{ the } \bar{a} \text{ constraint is binding on } i \textit{ or } \text{the } \underset{\sim}{a} \text{ constraint is binding on } j \text{ (or both).} \tag{9}$$

To verify (9), consider the pair of strategies (s_i, s_j) illustrated in Figure 1, which has the property that at $\theta^* = \arg\min\left(s_i(\cdot) - s_j(\cdot)\right)$, the $\bar{a}$ constraint is not binding on i and the $\underset{\sim}{a}$ constraint is not binding on j. The strategies depicted in the figure cannot form part of an MPE profile. To show this, we assume that s_j is a best response to $(s_i, \mathbf{s}_{-i,j})$, and conclude that s_i cannot be a best response to $(s_j, \mathbf{s}_{-i,j})$. Let $\Delta a = (s_i(\theta^*) - s_j(\theta^*))$ and consider player j's decision. Because t depends only on the average announcement $\mu(\mathbf{s})$, and since s_j is by assumption j's best response to $(s_i, \mathbf{s}_{-i,j})$, it follows that $s_j(\theta^*) + \Delta a = s_i(\theta^*)$ must be player-type (j, θ^*)'s best response to $(s_i - \Delta a, \mathbf{s}_{-i,j})$. But this observation implies that $s_i(\theta^*)$ cannot be (i, θ^*)'s best response to $(s_j, \mathbf{s}_{-i,j})$. To see why, note that since $k_i > k_j$, it follows from A5(ii) that against the *same* strategies, (i, θ^*)'s optimal response must strictly exceed (j,θ^*)'s: in particular, (i,θ^*)'s best response to $(s_i - \Delta a, \mathbf{s}_{-i,j})$ must strictly exceed (j,θ^*)'s, which is $s_i(\theta^*)$. Next, by definition of θ^*, $s_j(\cdot) \leq s_i(\cdot) - \Delta a$, so property (6) implies that (i,θ^*)'s best response to $(s_j, \mathbf{s}_{-i,j})$ must exceed his best response to $(s_i - \Delta a, \mathbf{s}_{-i,j})$, which, as we have shown, exceeds $s_i(\theta^*)$. Thus, s_i cannot be a best response to $(s_j, \mathbf{s}_{-i,j})$.

The first two parts of Prop. 2 follows almost immediately from (9). If either of the two constraints mentioned in (9) is satisfied, then $s_i(\theta^*) - s_j(\theta^*) \geq 0$. Since θ^* minimizes $\big(s_i(\cdot) - s_j(\cdot)\big)$, the function is nonnegative on its entire domain. Part i) of the proposition now follows immediately from the definitions in (1). Moreover, since neither player is constrained on $\big(\underset{\sim}{\theta}_j(\mathbf{s}), \tilde{\theta}_i(\mathbf{s})\big)$, property (9) implies that $\big(\underset{\sim}{\theta}_j(\mathbf{s}), \tilde{\theta}_i(\mathbf{s})\big)$ cannot be part of Θ^*, implying that on $\big(\underset{\sim}{\theta}_j(\mathbf{s}), \tilde{\theta}_i(\mathbf{s})\big)$, $s_i(\cdot) - s_j(\cdot) > s_i(\theta^*) - s_j(\theta^*) \geq 0$, establishing the strict inequality in part ii). To motivate the third part of the proposition, first note that since the domain of u is compact, all relevant derivative functions of u are uniformly continuous, and, if always non-zero, then they are bounded away from zero. Now suppose that there is a player-type (j,θ) whose first order condition, $U_j'(s_j(\theta), \theta; \mathbf{s}_{-j})$ is zero. For i with $k_i > k_j$ $U_i'(s_j(\theta), \theta; \mathbf{s}_{-i})$ exceeds $U_j'(s_j(\theta), \theta; \mathbf{s}_{-j})$ by an amount that is big oh of $1/n$.[13] Since $U_i'(\cdot,\cdot;\mathbf{s}_{-j})$ depends on i's type and announcement only through the mean type and mean announcement, the effects on $U_i'(\cdot,\cdot;\mathbf{s}_{-i})$ of i's announcement and hs type are big oh of $1/n^2$. Since A is compact, i's response is pushed to the upper edge of A as n increases without bound. The proof of the fourth part is analogous. An immediate implication of (9) is

Proposition 3 (At most one player is unconstrained): *In any MPE, at most one player's strategy is almost-never-constrained.*

To verify Prop. 3, observe from (9) that if i is not up-constrained at $\theta^* \in \Theta^*$, then j must be down-constrained. Since by definition Θ^* is nonempty, in equilibrium it can never happen that both s_i and s_j are almost-never-constrained. That is, regardless of the width of the announcement space A, an equilibrium cannot exist unless misreporting by all but at most one player increases to the extent that with positive probability, their announcements are constrained by one of the boundaries. Thus Prop. 3 highlights the role of the announcement bounds in ensuring the existence of MPE.

We conclude this section with a discussion of the class of strategies on which we will focus for the remainder of the paper. Letting $\iota(\cdot)$ denote the identity map on Θ, player r's strategy is said to be

---

[13] A function $f(x)$ is said to be big oh of $g(x)$ if there exists $M \in \mathbb{R}$ such that for all x, $|f(x)| < M|g(x)|$.

constrained unit affine (CUA) if for some $\lambda \in \mathbb{R}$, $s_r(\cdot) = \min\{\bar{a}, \max\{\underline{a}, \iota(\cdot) + \lambda\}\}$
unit affine if neither bound on the announcement space is binding, i.e., if $s_r(\cdot) = \iota(\cdot) + \lambda_r$

The defining property of a CUA strategy is that the extent of r's mis-representation of his type is independent of this type, except when r is constrained by the boundaries of A. The parameter λ_r indicates the extent of this mis-representation. A CUA strategy is unit affine iff it is also almost-never-constrained. CUA strategies are a special class of nondegenerate strategies that play an central role in our analysis. Next, note that the set of degenerate CUA strategies $\{s_r(\cdot) = \min\{\bar{a}, \max\{\underline{a}, \iota(\cdot) + \lambda\}\} : \lambda_r \leq \underline{a} - \overline{\theta}\}$ are all functionally equivalent: in each case, $s_r(\cdot) = \underline{a}$. Similarly all CUA strategies with $\lambda_r \geq \bar{a} - \underline{\theta}$ are equivalent. Hence we can impose without loss of generality (w.l.o.g.) that

$$s_r(\cdot) = \min\{\bar{a}, \max\{\underline{a}, \iota(\cdot) + \lambda\}\} \text{ is an \textit{admissible CUA strategy} iff } \lambda_r \in \Lambda \equiv [\underline{a} - \overline{\theta}, \bar{a} - \underline{\theta}]. \quad (10)$$

Since $\bar{a} > \underline{a}$ and $\overline{\theta} > \underline{\theta}$, the set Λ is nonempty. Observe from (1a) and (1b) that if s_r is CUA, then

$$\underset{\sim}{\theta}_r(s_r) \quad = \quad \min\{\underline{\theta}, \underline{a} - \lambda_r\} \quad < \quad \max\{\overline{\theta}, \bar{a} - \lambda_r\} \quad = \quad \tilde{\theta}_r(s_r). \quad (11)$$

If $\Theta \subseteq [\underline{a}, \bar{a}]$ we say that the announcement space is *inclusive*. It follows from (11) that

$$\text{if } \Theta \text{ is inclusive then no CUA strategy is bi-constrained} \quad (12)$$

To see this, note that if Θ is inclusive and $\lambda_r \geq 0$ then $s_r(\underline{\theta}) = \underline{\theta} + \lambda_r \geq \underline{a} + \lambda_r \geq \underline{a}$; similarly, if $\lambda_r \leq 0$ then $s_r(\overline{\theta}) \leq \bar{a}$,

4. Aggregation Games with Common Preferences

Assumption A1(ii) specifies that all players have distinct observable characteristics. For this section only, we reverse this assumption, and consider games in which players' observable characteristic are identical. We also assume that the announcement space is inclusive, so that truthful type revelation is feasible. This analysis will serve as a useful benchmark when we consider games in which players' observable characteristics are heterogeneous and when the bounds on the announcement space preclude complete truthful revelation. The analysis highlights the importance of unit affine strategies: we will show that in n-player games, there are equilibria—including one characterized by truthful type revelation—in which players' strategies are unit affine and satisfy a strong efficiency criterion. Moreover, in two-player games, equilibrium strategies are *necessarily* unit affine, and *all* equilibria satisfy this criterion.

We now introduce our notion of efficiency. An action $s_r(\theta)$ is a *best conceivable response* for player-type (r,θ) to $\mathbf{s}_{-r}$ if for all $\mathbf{s}'_{-r}$ and all $a \in A$, $U_r(s_r(\theta),\theta;\mathbf{s}_{-r}) \geq U_r(a,\theta;\mathbf{s}'_{-r})$. When a player-type's action is a best conceivable response to other players' strategies, this player's expected payoff could not be higher, even if he had total control over the strategies played by all other players! An MPE is now defined to be *efficient* if every player-type's action is a best conceivable response to other players strategies. This is clearly an extremely stringent notion of efficiency.

A strategy profile will be called *zero-sum unit affine (ZSUA)* if each player's strategy is unit affine and if there is truthful revelation in aggregate. Specifically, let $\mathbf{\Lambda} = \{\boldsymbol{\lambda} \in \Lambda^n : \sum_{r=1}^n \lambda_r = 0\}$. A strategy profile is ZSUA if for some $\boldsymbol{\lambda} \in \mathbf{\Lambda}$, $s_r = \theta_r + \boldsymbol{\lambda}_r$, for each r.[14] Given a profile $\mathbf{s}$, $\mu(\mathbf{s})$ is identically equal to $\mu(\boldsymbol{\theta})$ iff $\mathbf{s}$ is ZSUA; that is, ZSUA profiles truthfully reveal types in the aggregate and vice versa. A special case is when $\boldsymbol{\lambda} = 0$, i.e., each individual agent reveals his type. The following proposition highlights the intuitive fact that in an aggregation game, incentives for strategic behavior arise only when there are *ex ante* differences between agents' characteristics, i.e., their k's.

Proposition 4 (ZSUA profiles as equilibrium strategies): *Consider an inclusive aggregation game in which $k_r = \bar{k}$ for all r. A sufficient condition for a strategy profile to be an equilibrium is that it is ZSUA. Further, a ZSUA equilibrium is efficient.*

The proof of Prop. 4 is immediate. Consider $\boldsymbol{\lambda} = (\lambda_r, \lambda_{-r}) \in \mathbf{\Lambda}$. Necessarily, $\lambda_r = -\sum_{i\neq r} \lambda_r$. In the ZSUA strategy profile corresponding to $\boldsymbol{\lambda}$, player-type (r,θ) reports $s_r(\theta) = \theta + \lambda_r$. Consequently

$$U_r(\mathbf{s}_r(\theta),\theta;\mathbf{s}_{-r}) \quad = \quad \int_{\mathbf{\Theta}} u(t(\mathbf{s}(\boldsymbol{\vartheta}),\mathbf{k}),\boldsymbol{\vartheta},\bar{k})h(\boldsymbol{\vartheta})d\boldsymbol{\vartheta} \quad = \quad \int_{\mathbf{\Theta}} u(t(\boldsymbol{\vartheta},\mathbf{k}),\boldsymbol{\vartheta},\bar{k})h(\boldsymbol{\vartheta})d\boldsymbol{\vartheta}$$

Since players' observable characteristics are all identical, the social welfare function (defined in (2)) coincides with each player's utility function: $w(t,\boldsymbol{\theta},\mathbf{k}) = u(t,\boldsymbol{\theta},\bar{k})$. Since the outcome function is assumed to be CISE, we have $t = \text{argmax}\ u(\cdot,\boldsymbol{\theta},\bar{k})$ for every $\boldsymbol{\theta} \in \mathbf{\Theta}$, Thus, the ZSUA profile

[14]Clearly, for any vector $\boldsymbol{\lambda}$ with $\lambda_r < (\underline{a} - \underline{\theta})$ (or $\lambda_r > \bar{a} - \bar{\theta}$), $s_r = \theta_r + \boldsymbol{\lambda}_r$ would not be admissible for types in some neighborhood of $\underline{\theta}$ (or $\bar{\theta}$).

maximizes the expected utility of every player and constitutes an MPE. Further, since each player obtains the highest possible utility, the equilibrium is also efficient.

When there are only two players with identical observable characteristics, we can go much further. In this case, the preceding and following propositions establish that a profile is an equilibrium if and only if it is ZSUA, i.e., *all* equilibria are efficient![15]

Proposition 5 (MPE are ZSUA):† *Consider a two player inclusive aggregation game with $k_i = k_j$. A necessary condition for a strategy profile to be an MPE is that it is ZSUA.*

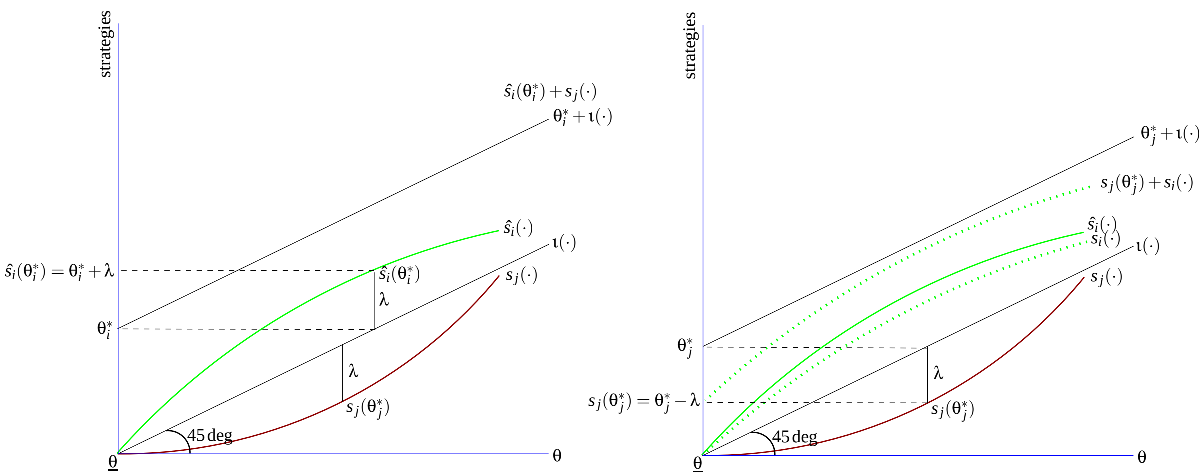

FIGURE 2. Intuition for Prop. 5

Figure 2 provides some intuition. Consider a strategy that is not unit affine, such as s_j in the left panel of the figure. Letting $\iota(\cdot)$ denote the identity map, the maximum value of $(\iota(\cdot) - s_j(\cdot))$ is λ, which is achieved uniquely at θ_j^*.[16] We first establish that a necessary condition for s_i to be a best response to s_j is that $(s_i(\cdot) - \iota(\cdot))$ is everywhere strictly less than λ. To see this, consider a strategy such as $\hat{s}_i$ satisfying, for some θ_i^*, $(\hat{s}_i(\theta_i^*) - \theta_i^*) \geq \lambda$. Given any such strategy for i, the aggregate strategy $\hat{s}_i(\theta_i^*) + s_j(\cdot)$—i.e., the highest curve in the left panel—must lie above the line $\theta_i^* + \iota(\cdot)$ with probability one. That is, for player-type (i, θ_i^*), the average of players' announced types exceeds the average of their actual types with probability one. Since $t(\cdot)$ is CISE and the social welfare function $w(\cdot)$ coincides with i's and j's common utility function, the outcome generated

[15] An immediate implication of the argument below is that when players' observable characteristics are identical and the announcement space coincides with Θ, then the *unique* equilibrium for a two-player aggregation game is that players truthfully reveal their private information with probability one.

[16] Uniqueness is not required, but it simplifies the intuitive exposition.

by $(s_j, \hat{s}_i)$ must be super-optimal for (i, θ_i^*) with probability one. Conclude that $\theta_i^* + \lambda$ is not a best response for (i, θ_i^*) against $s_j(\cdot)$; more generally, for s_i to be optimal against any not unit affine s_j, it is necessary that $(s_i(\cdot) - \iota(\cdot)) < \max(\iota(\cdot) - s_j(\cdot))$. Now consider any strategy satisfying this necessary condition—e.g., the dashed curve $s_i(\cdot)$ in the right panel—and observe that the aggregate strategy $s_j(\theta_j^*) + s_i(\cdot)$ is everywhere *below* the line $\theta_j^* + \iota(\cdot)$, and hence sub-optimal for (j, θ_j^*). We have shown, then, that the action $s_j(\theta_j^*)$ cannot be a best response for (j, θ_j^*), against *any* strategy that could possibly be a best response against the arbitrarily chosen, not unit affine $s_j(\cdot)$.

5. Games with Quadratic Payoff functions

In our introductory discussion in §1, our players reported to the center, who took an action, τ, that affected all of them. For the remainder of the paper, we abstract from the issue of how the center uses the information that players provide and assume, simply, that each player incurs a loss that is quadratic in the difference between that player's observable characteristic and the gap between the means of actual and reported information. Formally, we define the utility function for a player with observable characteristic k as the biased quadratic loss function:[17]

$$u(\tau, \mu(\boldsymbol{\theta}), k) \quad = \quad -(k + \mu(\boldsymbol{\theta}) - \tau)^2. \tag{13}$$

With this specification, the CISE property requires the center to average the types that players announce: $\tau = t(\mathbf{s}, \mathbf{k}) = \mu(\mathbf{s})$. A game with utilities given by (13) will be called a *quadratic aggregation game*. It is straightforward to verify that given t, (13) satisfies Assumptions A4-A6. The goal of a player with observable characteristic $k > 0$ is to induce the center to overestimate the value of $\mu(\boldsymbol{\theta})$ by an amount that is as close as possible to k. Specifically, the optimal expected outcome for a player with observable characteristic k_r and type parameter θ_r is $E_{\boldsymbol{\vartheta}_{-r}} t = k_r + E_{\boldsymbol{\vartheta}_{-r}} \mu(\langle \theta_r, \boldsymbol{\vartheta}_{-r} \rangle)$.

This quadratic specification is consistent with either of the two interpretations of our model proposed in §1. For the non-statistical interpretation, the relationship is self-evident: players lose utility with the square of the difference between the composite score implied by players' actual types, adjusted by the player's personal bias, and the score that the center would compute by aggregating players' announcements. Under the Bayesian interpretation, each player loses utility with the square of the difference between the posterior mean computed by the center from announcements and the one implied by actual types, again after adjusting for the player's bias. Under very general

[17] As noted in fn. 5, this specification is very widely used.

conditions, the posterior mean is an affine function of the sample mean.[18] If the posterior mean is defined as $b_0 + b_1\mu(\boldsymbol{\theta})$, the loss function implied by our Bayesian interpretation is

$$-\big(k + (b_0 + b_1\mu(\boldsymbol{\theta})) - (b_0 + b_1\mu(\mathbf{s}))\big)^2 \;=\; -\big(k + b_1(\mu(\boldsymbol{\theta}) - \mu(\mathbf{s}))\big)^2 \;=\; -\big(k + b_1(\mu(\boldsymbol{\theta}) - \tau)\big)^2.$$

By choosing appropriately the units of the vector $\mathbf{k}$, we can set $b_1 = 1$ and recover (13).

While this Bayesian interpretation is suggestive, there is a notable distinction between our quadratic loss function and the canonical Bayesian loss function. To best appreciate the difference, consider (13) for an unbiased player, i.e., set $k = 0$. Then the only source of loss is that players mis-report the signals they receive; our players are modeled as uninterested in the difference between the mean of their signals and the *true* mean of the distribution from which their signals were drawn. In the classical Bayesian problem, on the other hand, the latter difference is all that matters; the possibility of mis-reporting does not arise.

In most respects, this distinction is unimportant and our specification captures exactly what we are interested in, i.e., the information losses that arise because players are strategic and are constrained by the boundaries.[19] In one respect, however, the omitted difference is significant: in a game small enough to admit non-degenerate strategies, it does not capture the full welfare impact in a Bayesian setting of increasing *n*, since it ignores the welfare benefit of increasing the precision with which the aggregate signal estimates the true mean (i.e., reducing the second term in (14)). As an extreme example, when all players have the same observable characteristic as in §4, our players attain their first-best outcomes in every game, regardless of *n*; had we defined players' utility as a standard Bayesian loss function, the first-best would be approached only asymptotically.

5.1. **CUA strategies.** The quadratic specification ensures that equilibrium strategies will be CUA (see p 17). Given the utility (13) and outcome function $t(\mathbf{s},\mathbf{k}) = \mu(\mathbf{s})$, if r were not required to respect the admissibility bounds (10) on λ_r, his optimal response to $\mathbf{s}_{-r}$ would be the UA strategy

[18] Bernardo and Smith (2000, Proposition 5.7 (pp. 275-276)) establishes this for exponential families of distributions.

[19] For instance, if the θ_i's were independently drawn from a distribution with $E(\theta_i) = \theta^t$ and players' utility depended on the true mean θ^t rather than the average realized signal $\mu(\boldsymbol{\theta})$, the expected quadratic loss would be

$$-E(k + \theta^t - \tau)^2 \quad \equiv \quad -E_{\boldsymbol{\vartheta}}(k + \mu(\boldsymbol{\vartheta}) - \tau + \theta^t - \mu(\boldsymbol{\vartheta}))^2 \quad = \quad -E_{\boldsymbol{\vartheta}}(k + \mu(\boldsymbol{\vartheta}) - \tau)^2 - E_{\boldsymbol{\vartheta}}(\theta^t - \mu(\boldsymbol{\vartheta}))^2. \qquad (14)$$

The first part of the loss arises entirely due to misreporting and coincides with $u(\cdot)$ in (13); the second part, which is precisely the canonical Bayesian loss function, is omitted from our model.

$\theta_r + \lambda_r$, where

$$\lambda_r = nk_r + \sum_{i \neq r} E_{\vartheta_i}\left(\vartheta_i - s_i(\vartheta_i)\right). \tag{15}$$

In general, the UA response $\theta_r + \lambda_r$ will not belong to A for all values of θ_r, particularly if $|k_r|$ is large. Accordingly, r's *constrained* optimal response will be

$$s_r(\theta_r) = \min\{\bar{a}, \max\{\theta_r + \lambda_r, \underset{\sim}{a}\}\}. \tag{16}$$

To identify an NMPE, we need to compute the $\boldsymbol{\lambda}$ vector which solves the set of n equations in (15) subject to the constraint (16). As a first step, we let $\xi_r(\cdot)$ denote player r's *deviation from affine*, defined as the difference between the CUA strategy $s_r(\cdot)$ and the UA strategy $\iota(\cdot) + \lambda_r$. Given λ_r, let $E\xi_r$ denote r's *expected deviation from affine*:

$$E\xi_r \quad \equiv \quad E_{\vartheta_r}(s_r(\vartheta_r) - (\vartheta_r + \lambda_r)) \quad = \quad E_{\vartheta_r}\big(\min\{\bar{a}, \max\{\underset{\sim}{a}, \vartheta_r + \lambda_r\}\} - \vartheta_r\big) \; - \; \lambda_r \tag{17}$$

$$= \quad \int_{\underline{\theta}}^{\underset{\sim}{\theta}_r} (\underset{\sim}{\theta}_r - \vartheta_r) dH(\vartheta_r) \; + \; \int_{\tilde{\theta}_r}^{\bar{\theta}} (\tilde{\theta}_r - \vartheta_r) dH(\vartheta_r), \tag{18}$$

where, from (1a) and (1b), $\underset{\sim}{\theta}_r(\lambda_r) = \underset{\sim}{a} - \lambda_r$ and $\tilde{\theta}_r(\lambda_r) = \bar{a} - \lambda_r$. Thus $E\xi_r$ is a measure of the impact of the bounds $\bar{a}$ and $\underset{\sim}{a}$ on r's expected announcement. Clearly,

$$\text{if } r \text{ is single-constrained and } E\xi_r \neq 0, \; \lambda_r E\xi_r < 0. \tag{19}$$

Since we focus exclusively on CUA strategies in the remainder of the paper, we will sometimes use the symbol λ_r as a shorthand for the uniquely defined CUA strategy with parameter λ_r.

We note in passing two implications of (17) and (18) that we will use later. First, aggregating the identity in (17) across players and rearranging, we obtain

$$E_{\boldsymbol{\vartheta}}\big(\mu(\mathbf{s}^*(\boldsymbol{\vartheta})) \; - \; \mu(\boldsymbol{\vartheta})\big) \quad = \quad \mu(\boldsymbol{\lambda}^*) \; + \; \mu(E\boldsymbol{\xi}). \tag{20}$$

Second, differentiating (18) w.r.t. λ_r and inferring from (11) that $H(\underset{\sim}{\theta}_r) < H(\tilde{\theta}_r)$:

$$\frac{dE\xi_r}{d\lambda_r} \quad = \quad -\Big(H(\underset{\sim}{\theta}_r) \; + \; 1 \; - \; H(\tilde{\theta}_r)\Big) \quad \subset \quad (-1, 0] \tag{21}$$

$$\text{and} \quad \frac{dE\xi_r}{d\lambda_r} \quad = \quad 0 \qquad \text{iff } r \text{ is almost never constrained}$$

Substituting $\theta_i - s_i(\theta_i) = -(\lambda_i + \xi_i)$ into (15) and rearranging, it follows that if $\boldsymbol{\lambda}^*$ is an MPE,

$$nk_r \quad = \quad \sum_i \lambda_i^* + \sum_{i \neq r} E\xi_i(\lambda_i^*), \qquad \text{for all } r \text{ with } \lambda_r^* \in \text{int}(\Lambda). \tag{15$'$}$$

Figure 3 provides some intuition for (15′), for the simple game with two players i and j and $0 < k_i = -k_j$. The figure is a diagonal cross-section of the three-dimensional graph from $\Theta \times \Theta$ to outcomes, that is, the graph depicts the event that i and j observe the same private signals. Player i is up-constrained while player j is down-constrained. The thick kinked line represents

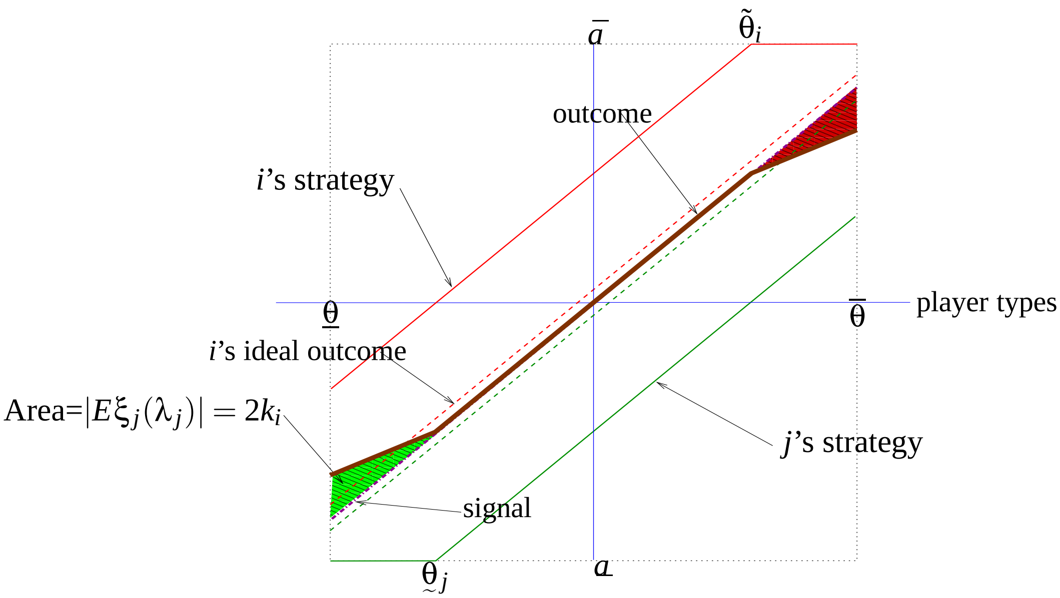

FIGURE 3. Intuition for display (15′)

the outcome as a function of type realizations, given the two players' strategies. The important property highlighted by the kinked line is that when $\theta_i > \tilde{\theta}_i$ (and $\theta_j \in [\underset{\sim}{\theta}_j, \tilde{\theta}_j]$), the realized outcome is an *under*-estimate of the realized type, while when $\theta_j < \underset{\sim}{\theta}_j$ (and $\theta_i \in [\underset{\sim}{\theta}_i, \tilde{\theta}_i]$), it is an *over*-estimate; when $\theta_r \in [\underset{\sim}{\theta}_r, \tilde{\theta}_r]$, for $r = i, j$, the outcome accurately reflects the aggregate signal. Now consider the outcome from player i's perspective and for concreteness, suppose $\theta_i = 0$ and the horizontal axis represents j's type. Player i's *ex poste* ideal outcome, as a function of j's type, is represented by the dashed line above the diagonal: for every value of j's type, i's *ex poste* ideal outcome exceeds it by k_i. When j is unconstrained, his under-report exactly counteracts i's over-report, resulting in an outcome that is suboptimal from i's perspective; however, at low values of θ_j, the constraint $\underline{a}$ binds j's under-reporting, resulting in an outcome exceeding i's ideal outcome. Equation (15′) describes how the over- and under-estimates are balanced in equilibrium: the expected over-estimate of the true average equals twice the expected under-estimate.

The following, immediate implication of (15′) will prove very useful in what follows. If $\boldsymbol{\lambda}^*$ is an MPE, then for all i, j with $\lambda_i^*, \lambda_j^* \in \mathrm{int}(\Lambda)$,

$$n(k_i - k_j) \quad = \quad E\xi_j(\lambda_j^*) \; - \; E\xi_i(\lambda_i^*). \tag{22}$$

To motivate (22), suppose $k_i > k_j$ and both i and j are up-constrained. From part ii) of Prop. 2, $k_i > k_j$ implies $\lambda_i > \lambda_j$, so the constraint $\bar{a}$ binds more tightly on i than on j, i.e., $E\xi_i < E\xi_j$.

Proposition 6 (Uniqueness of MPE):† *Every quadratic aggregation game has a unique MPE.*

5.2. **MPE outcomes and payoffs.** The quadratic setup allows us to analyze each player's equilibrium performance: to what degree the outcome of the game matches his ideal outcome, and how his payoff depends on player characteristics. We begin by introducing a notion describing the degree to which each player "gets what he wants" in equilibrium. We define as a benchmark the *complete information personally optimal (CIPO) outcome* for player r: this outcome would maximize r's payoff if he had complete information about the average type. We denote this "ideal" outcome from r's perspective by $\hat{t}(\boldsymbol{\theta}, k_r)$. From (13), r's CIPO outcome is

$$\hat{t}(\boldsymbol{\theta}, k_r) \quad = \quad \mu(\boldsymbol{\theta}) + k_r. \tag{23}$$

If $\mu(\mathbf{s}^*)$ is the equilibrium outcome of the game. then the difference $E_{\boldsymbol{\vartheta}}\left(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))\right) - \hat{t}(\boldsymbol{\vartheta}, k_r)\big)$, which we label as r's *expected CIPO deviation*, is a measure of the degree to which the equilibrium outcome differs in expectation from player r's CIPO outcome. Prop. 7 below establishes that in an NMPE, the expected CIPO deviation is $1/n$ times the size of the player's expected deviation from affine. This result is striking because the latter depends only on r's strategic choice, while the former depends on *all* players' choices. Note also from (19) that a single-constrained player who over- (under-) reports his type can expect a sub- (super-) optimal outcome.

Proposition 7 (The expected CIPO deviation):† *If $\mathbf{s}^* = \boldsymbol{\theta} + \boldsymbol{\lambda}^*$ is an MPE profile of a quadratic aggregation game, and $\lambda_r^* \in int(\Lambda)$, then r's expected CIPO deviation is $E\xi_r(\lambda_r^*)/n$.*

Since the expected deviation from affine measures how tightly the announcement bounds restrict r's action in equilibrium, Prop. 7 indicates that a player whose action is more restricted is less likely to obtain his CIPO outcome in expectation.

After r learns his type θ_r, a parallel measure of deviation from his ideal outcome is the *interim expected CIPO deviation*, defined as the difference $E_{\boldsymbol{\vartheta}_{-r}}\left(\mu(\langle s^*_r(\theta_r), \mathbf{s}^*_{-r}(\boldsymbol{\vartheta}_{-r})\rangle) - \hat{t}(\langle\theta_r, \boldsymbol{\vartheta}_{-r}\rangle, k_r)\right)$, where $E_{\boldsymbol{\vartheta}_{-r}}\hat{t}(\langle\theta_r, \boldsymbol{\vartheta}_{-r}\rangle, k_r) = E_{\boldsymbol{\vartheta}_{-r}}\mu(\langle\theta_r, \boldsymbol{\vartheta}_{-r}\rangle) + k_r$ is r's *interim expected CIPO outcome*. Similar to Prop. 7, Prop. 8 establishes that r's interim expected CIPO outcome is implemented in equilibrium if and only if his strategy is unconstrained at θ_r:

Proposition 8 (Interim Implementation):† *For a player r of type θ_r, his interim expected CIPO deviation equals zero, or his interim expected CIPO outcome is implemented in equilibrium, if and only if his strategy s^*_r is unconstrained at θ_r.*

The previous discussion indicates that player r's expected deviation from affine i.e., the expected degree to which r's strategies are restricted by the announcement bounds, is instrumental in determining whether r gets "what he wants." We next illustrate how the deviation from affine affects a player's expected equilibrium payoff. From (13), r's expected payoff from a strategy profile $\boldsymbol{\lambda}$ is $-E_{\boldsymbol{\vartheta}}\left(\mu(\boldsymbol{\vartheta}) + k_r - \mu(\mathbf{s})\right)^2$, i.e., the expectation of the squared difference between r's CIPO outcome and the realized outcome. For an arbitrary profile $\boldsymbol{\lambda}$, the expression for this expectation is exceedingly messy, reflecting the complexity of the interactions between multiple players' deviations from affine: in some regions of $\boldsymbol{\Theta}$, the distortion resulting from different players' constraints offset each other; in others they are mutually reinforcing. In equilibrium, however, *all* of these interaction terms disappear, leaving only the first and second moments of players' deviations from affine. Specifically, let $V\xi_r(\lambda_r)$ denote the (*ex ante*) variance of r's deviation from affine, i.e.,

$$V\xi_r(\lambda_r) = \mathrm{Var}_{\vartheta}\left(s_r(\vartheta_r) - (\vartheta_r + \lambda_r)\right) \tag{24}$$

Note that $V\xi_r(\lambda_r)$ depends only on r's *own* type realization. We now have:

Proposition 9 (Equilibrium Payoffs):† *Let $\mathbf{s}^* = \boldsymbol{\theta} + \boldsymbol{\lambda}^*$ be an MPE profile of a quadratic aggregation game. For each player r with $\lambda^*_r \in int(\Lambda)$, r's expected equilibrium payoff is*

$$E_{\boldsymbol{\vartheta}}u(\mu(\mathbf{s}^*), \mu(\boldsymbol{\vartheta}), k_r) \;=\; -E_{\boldsymbol{\vartheta}}\left(\mu(\boldsymbol{\vartheta}) + k_r - \mu(\mathbf{s}^*)\right)^2 \;=\; -\left(\mu(\mathbf{V}\boldsymbol{\xi}(\boldsymbol{\lambda}^*))/n \;+\; (E\xi_r(\lambda^*_r)/n)^2\right). \tag{25}$$

Prop. 7 and Prop. 9 are complementary. Prop. 7 established that player r's expected CIPO deviation coincides with his expected deviation from affine, deflated by n. But equilibrium expected payoffs depend on *squared* deviations from affine. Prop. 9 shows that players' expected payoffs are equally negatively impacted by the variances of each others' deviation from affine; the sole factor distinguishing two players' expected payoffs is the difference between their squared expected CIPO deviations.

We next study the aggregate equilibrium payoff of the players. From a normative perspective, there are two benchmark measures of welfare that we might consider. The more obvious is the average of players' equilibrium expected payoffs. We refer to this as *average private welfare*, defined as

$$\text{APW} \quad = \quad \frac{1}{n}E_{\boldsymbol{\vartheta}}\left(\sum_{i=1}^{n} u(\mu(\mathbf{s}), \mu(\boldsymbol{\vartheta}), k_i)\right). \tag{26}$$

Alternatively, one could take the view that *social* welfare should be evaluated from an *unbiased* perspective, i.e., from the perspective of a player whose observable characteristic is zero, reflecting a preference for truthful revelation. Accordingly we define *unbiased social welfare* as

$$\text{USW} \quad = \quad E_{\boldsymbol{\vartheta}} u(\mu(\mathbf{s}), \mu(\boldsymbol{\vartheta}), 0) \quad = \quad E_{\boldsymbol{\vartheta}}\left(\mu(\boldsymbol{\vartheta}) - \mu(\mathbf{s})\right)^2. \tag{27}$$

Our assumption that $\sum_i k_i = 0$ implies that APW and USW differ only by a constant. Specifically:

$$\begin{aligned} \text{APW} \quad &= \quad -\frac{1}{n}E_{\boldsymbol{\vartheta}}\sum_{i\in I}\left(\mu(\boldsymbol{\vartheta}) + k_i - \mu(\mathbf{s})\right)^2 \\ &= \quad -\frac{1}{n}\left\{\sum_i k_i^2 \quad + \quad 2\sum_i k_i E_{\boldsymbol{\vartheta}}\left(\mu(\boldsymbol{\vartheta}) - \mu(\mathbf{s})\right) \quad + \quad nE_{\boldsymbol{\vartheta}}\left(\mu(\boldsymbol{\vartheta}) - \mu(\mathbf{s})\right)^2\right\} \\ &= \quad \text{USW} \quad + \quad \sum_i k_i^2/n \end{aligned}$$

From (47) the following result is immediate.

Proposition 10 (Unbiased Social Welfare): *If $\boldsymbol{\lambda}^*$ is an MPE profile of a quadratic aggregation game, then unbiased social welfare is given by*

$$USW \quad = \quad -\left\{\mu(V\boldsymbol{\xi}(\boldsymbol{\lambda}^*))/n + \left(\mu(E\boldsymbol{\xi}(\boldsymbol{\lambda}^*)) + \mu(\boldsymbol{\lambda}^*)\right)^2\right\} \tag{28}$$

5.3. **Anchored Games.** The discussion so far illustrates the central role the expected deviation from affine plays in affecting equilibrium payoffs. There is a class of games in which some player j's expected deviation from affine is zero. This property holds if either j's strategy is never constrained or if the constraints on j associated with the two announcement bounds cancel each other out in expectation. We later study games in which such a j always exists: in §6, j is the "middle" player in a symmetric game; in §7, the "largest" player in a game in which $\bar{a}$ never binds. We refer to such a game as an *anchored game* and to player j as *the anchor*. An anchored game exhibits strong properties and is particularly easy to analyze. Since anchoredness is defined in terms of an equilibrium property—whether or not some player's expected deviation from affine is zero—we must first prove that a game in a certain class has a unique MPE that exhibits this property, before invoking the properties of anchored games identified in Prop. 11 and Prop. 12.

Proposition 11 (Properties of anchored games): *Let $\boldsymbol{\lambda}^*$ be an MPE profile of an anchored quadratic aggregation game and let j be the anchor. For each player r with $\lambda^*_r \in int(\Lambda)$,*

i) *r's expected deviation from affine is $n(k_j - k_r)$,*

ii) *r's expected CIPO deviation is $(k_j - k_r)$,*

Part i) is obtained by combining (22) with the defining property of an anchored game, i.e., $E\xi_j(\lambda^*_j) = 0$. Part ii) then follows from Prop. 7. Strikingly, r's expected CIPO deviation depends exclusively on the gap between j's observable characteristic and r's, while r's expected deviation from affine depends both on this gap and n. To see why the latter is proportional to n, recall that r's objective is to shift the mean announcement by a magnitude k_r that is independent of n; the greater is n, the smaller is r's contribution to the mean, and hence the more must r mis-report. Note that the more r mis-reports, the more likely it is that he will be constrained by the announcement bounds. To study anchored games, we add assumption A7 to A1-A6. Parts (i) and (ii) simplify our anlysis. Part (iii) ensures that every anchored game has an NMPE.

Assumption A7: (i) The announcement space is inclusive (cf. p. 18); (ii) the type distribution is uniform with density parameter $h = 1/(\bar{\theta} - \underline{\theta})$; (iii) $||\mathbf{k}||_\infty < (\bar{\theta} - \underline{\theta})/4n$.

The combination of a quadratic loss function (13) and a uniform distribution over types (A7(ii)) is very widely used.[20] A7(iii) guarantees that our MPE is non-degenerate: it is needed because if the k's are far apart and n is sufficiently large, outlying players, attempting to steer the average announcement in their favor, will choose strategies that are constrained with probability one by one of the announcement bounds. To verify that A7(iii) guarantees that this will not happen, it suffices to check that $E\xi_r(\lambda^*_r) = n(k_j - k_r)$ is consistent with $\lambda^*_r \in \mathrm{int}(\Lambda)$. Assuming w.l.o.g. that $\lambda^*_r > 0$,

[20] See Crawford and Sobel (1982), Gilligan and Krehbiel (1989), Krishna and Morgan (2001), Morgan and Stocken (2008), and many others.

(18) implies:

$$E\xi_r(\lambda_r^*) \quad = \quad \int_{\bar{a}-\lambda_r^*}^{\bar{\theta}} (\vartheta_r + \lambda_r^* - \bar{a})\, dh(\vartheta_r) \quad = \quad 0.5h\left(\lambda_r^* + \bar{\theta} - \bar{a}\right)^2$$

$$\text{so that} \quad \lambda_r^* + \bar{\theta} - \bar{a} \quad = \quad \sqrt{\frac{2}{h} E\xi_r} \quad = \quad \sqrt{\frac{2}{h} n(k_j - k_r)} \qquad \text{if } \lambda_r^* \in \text{int}(\Lambda). \tag{29}$$

The last equality follows from Prop. 11. Also, from A7(iii),

$$2n(k_j - k_r)/h \quad \leq \quad \frac{4n}{h}||\mathbf{k}||_\infty \quad < \quad \frac{4n}{h}(\bar{\theta} - \underline{\theta})/4n \quad = \quad (\bar{\theta} - \underline{\theta})^2$$

so that $\lambda_r = \sqrt{\frac{2}{h}n(k_j - k_r)} - (\bar{\theta} - \bar{a}) \; < \; \left((\bar{\theta} - \underline{\theta}) - (\bar{\theta} - \bar{a})\right) = \bar{a} - \underline{\theta}$, verifying that $\lambda \in \text{int}(\Lambda)$. For anchored games satisfying A7, we obtain a closed-form expression for equilibrium payoffs.

Proposition 12 (Equilibrium Payoffs in Anchored Games):† *Let j be the anchor in an anchored quadratic game satisfying A7. Then player r's expected payoff in an NMPE is*

$$\begin{aligned} E_{\boldsymbol{\vartheta}} u(\mu(\mathbf{s}^*), \mu(\boldsymbol{\vartheta}), k_r) \quad &= \quad -\left\{ \sum_{i=1}^{n} (k_j - k_i)^2 \left(\sqrt{\frac{8}{9nh|k_j - k_i|}} - 1 \right) + (k_r - k_j)^2 \right\} \\ &\leq \quad -\left\{ \sum_{i=1}^{n} (k_j - k_i)^2/3 + (k_r - k_j)^2 \right\}. \end{aligned} \tag{30}$$

Expr. (30) thus establishes an upper bound on expected payoffs that declines with n. As shall see below, however, this does *not* imply that expected payoffs themselves decline monotonically. In the following sections, we study how the equilibrium outcome and aggregate welfare are related to primitives of the game such as the vector $\mathbf{k}$ and the bounds $\bar{a}$ and $\underline{a}$. For an arbitrary quadratic game, it is impossible to obtain closed-form expressions for these effects. Accordingly, we will focus on two special classes of anchored games for which closed-form results can be obtained.

6. Symmetric games

In this section we study games which are symmetric in a strong sense. We say that the observable characteristic vector is symmetric if for every player $\bar{r}$ with $k_{\bar{r}} > 0$, there exists a *matched player* $\underline{r}$ with $k_{\underline{r}} = -k_{\bar{r}}$. There may in addition be one more, *middle* player m with $k_m = 0$. In §6.4 below, we will refer to players whose observable characteristics are positive (resp. negative) as the *right-wing (resp. left-wing) faction*. We say that the announcement space is symmetric if the announcement bounds $\underline{a}$ and $\bar{a}$ are symmetric about zero, i.e., if $\underline{a} = -\bar{a}$; finally, we say that the type distribution is symmetric if $\underline{\theta} = -\overline{\theta}$ and if θ is symmetrically distributed around its mean zero. We now say that a game is *symmetric* if all these conditions are satisfied. We will refer to a game satisfying A1-A7 as a *symmetric quadratic aggregation game* (SQAG).

Proposition 13 (NMPE of Symmetric Games):† *Every SQAG has a unique NMPE satisfying:* $E\xi_r(\lambda_r^*) = -nk_r$*, for all* r *with* $\lambda_r \in int(\Lambda)$*. Moreover,*

i) for each player $\overline{r}$ *and matched player* $\underline{r}$, $\lambda_{\underline{r}}^* = -\lambda_{\overline{r}}^*$;

ii) if there is a middle player m*, then* $\lambda_m^* = 0$.

The middle player, if there is one, is the only player who announces truthfully in equilibrium. Any other player always mis-announces and his expected deviation from affine is determined entirely by his observable characteristic and n. Symmetric games with a middle player are also anchored games (see §5.3). It is clear from Props. 13 and 7, however, that symmetric games without a middle player exhibit the same properties as those that have one. To streamline the exposition, we shall in the remainder of the section treat *all* symmetric games as if they were anchored.

It is immediate from Props. 9 and 13 that r's equilibrium expected payoff is entirely determined by k_r and the average of the second moments of all players' deviations from affine.

$$E_{\boldsymbol{\vartheta}} u(\mu(\mathbf{s}^*), \mu(\boldsymbol{\vartheta}), k_r) \;=\; -\left(\mu(V\boldsymbol{\xi})/n \,+\, k_r^2\right) \;=\; -\left\{\sum_i k_i^2\left(\sqrt{\frac{8}{9nh|k_i|}} \,-\, 1\right) + k_r^2\right\}. \qquad (25')$$

The second equality is obtained by substituting zero for k_j in (30). (27) and (25′) now yield an expression for unbiased social welfare:

$$\text{USW} \quad = \quad -\sum_i k_i^2\left(\sqrt{\frac{8}{9nh|k_i|}} \,-\, 1\right) \qquad (31)$$

Prop. 13 provides us with a powerful tool for analyzing and comparing the welfare properties of aggregation games with different parameters. The three parameters we study in the remainder of this section are: the number of players (§6.1); the magnitude of the bound on the announcement space (§6.2); and the heterogeneity of players' observable characteristics (§6.3). Throughout this section, whenever we make a statement relating to either $\overline{\theta}$, $\overline{a}$, or $k_{\overline{r}}$, we will be implicitly making as well the matching statement about $\underline{\theta}$, $\underline{a}$, or $k_{\underline{r}}$. In particular, when we study the effect of increasing $\overline{a}$, we will be simultaneously, but implicitly, reducing $\underline{a}$ to preserve symmetry.

6.1. **Effects of changing the number of players.** Since symmetric games are anchored, at least some of the impacts of changing n are straightforward: a player's strategy (although not his payoff) depends only on n and his own observable characteristic. From Prop. 11, a player's expected deviation from affine is proportional to n, while his expected CIPO deviation is independent of n: as n increases, each player except the middle one mis-reports to an increasing extent, while in equilibrium the net expected effect of players' distortions on the center's decision is unchanged. The effects of n on expected payoffs and welfare are more complex. While in general there is no closed-form expression for $V\xi_r$, A7 allows us to obtain determinate results. We will compare expected payoffs and welfare for a finite sequence of "comparable" games with more and more players. To make the games comparable, we relax assumption A1(ii) for the remainder of §6.1 and construct our sequence by cloning m times a base game with q players and observable characteristic vector $\boldsymbol{\kappa}$.[21] To ensure that A7(iii) is satisfied, we require that $m \leq M = \lfloor 1/(4qh||\boldsymbol{\kappa}||_\infty) \rfloor$.[22] Now consider the aggregate welfare $\mathrm{USW}(m)$ in the m'th game. Since from (31) and (25′), the difference between $\mathrm{USW}(m)$ and player r's expected payoff is independent of m, the comparative statics results we obtain for welfare apply also to payoffs. Rewriting (31):

$$\mathrm{USW}(m) = -m\sum_{i=1}^{q}\kappa_i^2\left(\sqrt{\frac{8}{9mqh|\kappa_i|}}-1\right) = \sum_{i=1}^{q}\left(m\kappa_i^2-\sqrt{\frac{8m|\kappa_i|^3}{9qh}}\right) \qquad (31')$$

If m were a real number rather than an integer, USW would be convex in m, with

$$\frac{d\mathrm{USW}}{dm} = \sum_{i=1}^{q}\left(\kappa_i^2-\sqrt{2|\kappa_i|^3/(9mqh)}\right) \qquad (32)$$

The i'th element of the summation is $\gtreqless 0$ as $|\kappa_i| \gtreqless 2/(9mqh)$. Let $M' = \max\{m \leq M : ||\boldsymbol{\kappa}||_\infty < 2/(9mqh)\}$ and $M'' = \max\{m \in \mathbb{N} : |\kappa_i| < 2/(9mqh), \forall i\}$; If hM is sufficiently small, $M' < M$ (since $2/9 < 1/4$) while if, in addition, $\max_i|\kappa_i| - \min_i|\kappa_i|$ is sufficiently small, $(M'', M]$ will be non-empty. Clearly, $\mathrm{USW}(\cdot)$ is strictly decreasing on $[1, M')$ and strictly increasing on $(M'', M]$.

[21]The argument below could be made rigorous without violating assumption A1(ii): simply clone as we propose, and then perturb the cloned vector slightly to ensure uniqueness while preserving symmetry. In our view, the loss of rigor involved in our approach is justified by the gain in parsimony.

[22]For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. For $m \leq M$, the game with m clones of $\boldsymbol{\kappa}$ has $n = mq$ players. so that $||\boldsymbol{\kappa}||_\infty \leq 1/(4mqh) = (\overline{\theta} - \underline{\theta})/4n$, verifying that A7(iii) is satisfied.

These results reflect the tension between two effects as m increases. The first is that players need to mis-report more to accomplish the same expected CIPO deviation; this lowers welfare. The second effect reflects the law of large numbers: as m increases, it becomes increasingly likely that players' deviations from the mean of the type distribution will offset each other, and hence their individual deviations from affine will be mutually offsetting also. Prop. 14 summarizes:

Proposition 14 (Comparative statics w.r.t. n): *In the unique NMPE of an SQAG:*

- *i)* *each original player's expected deviation from affine is proportional to n.*
- *ii)* *each original player's expected CIPO deviation is independent of n*

For a finite sequence of games obtained by cloning m times a vector $\boldsymbol{\kappa} \in K^q$;

- *i)* *USW and expected payoffs are convex with respect to the number of clones*
- *ii)* *Suppose $||\boldsymbol{\kappa}||_\infty < (\bar{\theta} - \underline{\theta})/4n$, i.e., the players are relatively homogenous in their observable characteristics. Then USW and player expected payoffs initially decrease, and then may increase, with the number of clones.*

To reiterate, these results should be evaluated in the context of the non-statistical interpretation of our model, rather than the Bayesian one (see p. 1 and p. 22).

6.2. **Effects of changing the announcement bounds.** From Prop. 13, player r's expected deviation from affine, $E\xi_r(\cdot)$, is independent of the announcement bound $\bar{a}$. If $k_r \neq 0$, expression (18) then implies that as $\bar{a}$ changes, λ_r must adjust so that $E\xi_r(\cdot)$ remains equal to nk_r. Specifically:

Proposition 15 (Effects of changing $\bar{a}$):† *In the unique NMPE of an SQAG:*

$$\frac{d\lambda_r}{d\bar{a}} = \begin{cases} 0 & \textit{if } r \textit{ is the middle player} \\ 1 & \textit{if } r \textit{ is up-constrained} \\ -1 & \textit{if } r \textit{ is down-constrained} \\ \frac{(1-H(\tilde{\theta}_r))-H(\underset{\sim}{\theta}_r)}{H(\underset{\sim}{\theta}_r)+(1-H(\tilde{\theta}_r))} & \textit{if } r \textit{ is bi-constrained} \end{cases} \tag{33}$$

When r is bi-constrained, the denominator of $\frac{d\lambda_r}{d\bar{a}}$ is the probability that r is constrained by at least one of the announcement bounds. The numerator is the difference between the probabilities that r is up- and down-constrained. If r is single-constrained, he increases the degree of his misreporting at exactly the rate that the bounds are relaxed; he responds more slowly if he is bi-constrained.

We now consider the welfare effect of a marginal change in the announcement bound. First note that if the announcement space is inclusive, no player will be bi-constrained in equilibrium. Then players with $k \neq 0$ will adjust their announcements to fully compensate for any change in the announcement bounds. Hence, players' utilities, as well as aggregate welfare, will be unaffected by any change in the bounds. Specifically, recall from Prop. 9 that r's expected payoff depends on the first moment of r's own deviation from affine, as well as the second moments of all players' deviations. If there is a middle player j, $\xi_j = 0$ always and thus $E\xi_j$ and $V\xi_j$ are unaffected by changes in $\bar{a}$. For any other player r, since the change in λ_r fully compensates the change in $\bar{a}$, the deviation from affine ξ_i (or its distribution) remains unchanged, so do its first and second moments.

This independence property no longer holds when at least one player's equilibrium strategy is bi-constrained. For some intuition for this difference, Figure 4 considers the impact of relaxing the announcement bounds, when the only bi-constrained player is the middle player, m. Whenever m's type lies outside the interval $[\underline{a}, \bar{a}]$, obliging him to mis-report his type, all players are negatively impacted. The areas of the large triangles at either end of the type spectrum indicate the magnitude of the distortion. When the bounds are relaxed to $[\underline{a}', \bar{a}']$, the sizes of these triangles shrink, reflecting a decline in the variance of m's deviation from affine. Ex ante, this change benefits all players equally, since, from Prop. 9, each player's payoff is decreasing in the *total* variances of all players. Prop. 16 provides an expression for the rate at which a bi-constrained player's variance declines with a relaxation of the bounds. The more players are initially bi-constrained, the greater is the collective benefit of a relaxation.

Proposition 16 (Effects of increasing the announcement bound $\bar{a}$):† *In the unique NMPE of an SQAG, as the announcement space expands:*

- *i)* *if initially the announcement space is inclusive, the equilibrium expected payoff of every player remains constant;*
- *ii)* *if initially some player is bi-constrained, then each player's equilibrium expected payoff is equally positively affected, as is unbiased social welfare. Specifically, letting I^* denote the set of players who are bi-constrained in equilibrium, player r's expected payoff increases by $-\frac{1}{n^2}\sum_{i\in I^*}\frac{dV\xi_i}{d\bar{a}}$, where*

$$\frac{dV\xi_i}{d\bar{a}} = \frac{4}{H(\underset{\sim}{\theta}_i) + (1 - H(\tilde{\theta}_i))} \times \left\{ (1 - H(\tilde{\theta}_i)) \int_{\underline{\theta}}^{\underset{\sim}{\theta}_i} (\vartheta_i - \underset{\sim}{\theta}_i) dH(\vartheta_i) \; - \; H(\underset{\sim}{\theta}_i) \int_{\tilde{\theta}_i}^{\bar{\theta}} (\vartheta_i - \tilde{\theta}_i) dH(\vartheta_i) \right\} < 0 \tag{34}$$

Prop. 16 delivers a clear policy message, at least in the context of symmetric games. Recall from (12) that a necessary condition for a player to be bi-constrained is that the type space is not a subset of the announcement space. When, as in the present paper, the type space is known by the policy-maker who sets the announcement bounds, it is Pareto optimal to select an announcement space large enough to contain the type space. More generally, of course, the bounds on the type space will not be known with certainty. In this case, since it is costless to expand the announcement space, and possibly costly to contract it, the announcement space should be as large as possible.

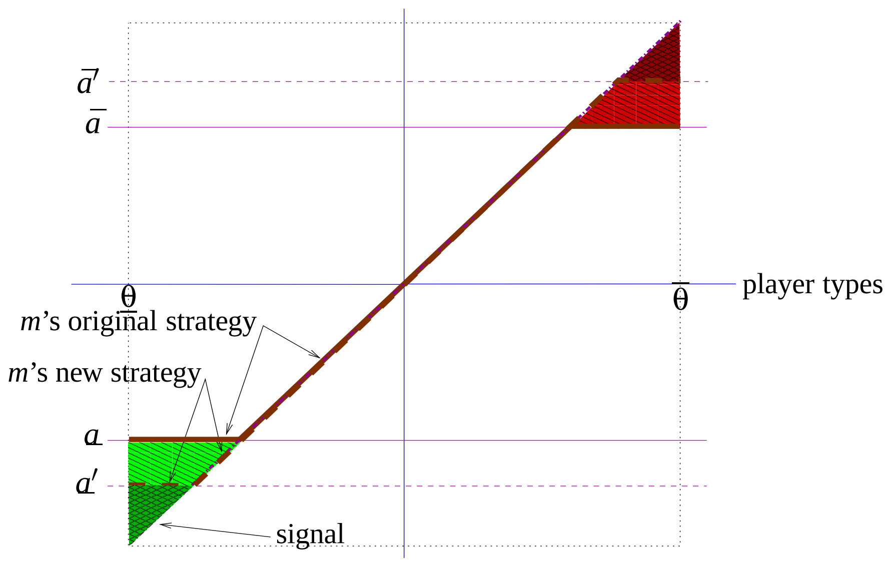

FIGURE 4. Intuition for Prop. 16

6.3. **Effects of increasing player heterogeneity.** In this subsection we study the impacts on the equilibrium outcome and on welfare of changes in the vector $\mathbf{k}$ of observable characteristics. Totally differentiating the identity $E\xi_r = -nk_r$ in (50) w.r.t. k_r and λ_r, we obtain

$$\frac{d\lambda_r}{dk_r} = \frac{n}{H(\underset{\sim}{\theta}_r) + (1 - H(\tilde{\theta}_r))} > n \tag{35}$$

where the denominator equals the probability with which player r is constrained by the announcement bounds. Thus, as k_r increases, λ_r also increases, and at a faster rate, to maintain the equilibrium property that $E\xi_r = -nk_r$. From Prop. 7 we know that as k_r increases and thus $|E\xi_r|$ increases, the difference between the expected equilibrium outcome and r's expected CIPO outcome also increases. Consequently r's expected payoff decreases. Prop. 17 quantifies this reduction:

Proposition 17 (Effects of dispersing players' observable characteristics):† *In the unique NMPE of an SQAG, if $k_r \neq 0$, then*

$$\frac{dV\xi_r}{d|k_r|} \quad = \quad 2n^2|k_r|\left(\frac{1}{H(\underset{\sim}{\theta}_r)+(1-H(\tilde{\theta}_r))}-1\right) \quad > \quad 0 \tag{36}$$

To see the effect of increasing $k_{\bar{r}}$ on players' expected payoffs, we totally derive the right hand side of (25) w.r.t. $k_{\bar{r}}$, noting that to preserve symmetry, $\frac{dk_{\underline{r}}}{dk_{\bar{r}}} = -1$, where $\underline{r}$ is $\bar{r}$'s matched player. As $k_{\bar{r}}$ increases, $\bar{r}$'s and $\underline{r}$'s welfare decline by $\left(\frac{2}{n^2}\frac{dV\xi_{\bar{r}}}{dk_{\bar{r}}}+2k_{\bar{r}}\right)$; for other players, the decline is $\frac{2}{n^2}\frac{dV\xi_{\bar{r}}}{dk_{\bar{r}}}$.

6.4. **Effects of increasing inter-faction player heterogeneity.** The results in §6.3 are hardly surprising: as players become more heterogeneous, the extent of their mis-reporting increases and this reduces welfare. The impact of an increase in *inter-faction* heterogeneity is less obvious. To explore this issue, we will reduce notation by assuming, in this subsection only:

Assumption A8: (i) $[\underline{\theta},\bar{\theta}] = [-1,1]$, so that $h(\cdot) = 1/2$; (ii) there is no middle player, so that each faction has $n/2$ players; (iii) n is divisible by 4.

Let $\bar{\mathbf{k}}^{+} \in (0,1)^{n/2}$ be a strictly increasing vector, denoting the observable characteristics of the right-wing faction.[23]. Pick a vector $\boldsymbol{\alpha} \in \mathbb{R}^{n/4}_{++}$ and let $\boldsymbol{dk} = (-\boldsymbol{\alpha},\boldsymbol{\alpha}) \in \mathbb{R}^{n/2}$. We will consider a family of right-wing profiles of the form $\{\bar{\mathbf{k}}^{+} + \gamma\boldsymbol{dk} : \gamma \approx 0\}$. The observable characteristics of the left-wing faction are implied by symmetry. An increase in the nonnegative scalar γ represents a faction-mean-preserving spread of each faction's profile of observable characteristics. As γ increases in a neighborhood of zero,[24] the moderate members of the faction become more moderate—the dk's are negative for the first $n/4$ faction members, all of whom have k's below the faction's median—while the extreme members become more extreme. Prop. 18 below establishes the following impacts of such a spread: if players' characteristics are initially quite homogeneous—specifically, contained in the interval $(-1/4n, 1/4n)$—the spread will reduce both USW and APW. If the factions are initially quite polarized—specifically, no player's characteristic belongs to $[-1/4n, 1/4n]$—the spread will increase USW (though not necessarily APW).

[23] Recall from p. 29 that player r belongs to the right-wing (resp. left-wing) faction if $k_r > 0$ (resp. $k_r < 0$)

[24] We need to keep γ close to zero to ensure that the perturbed vector $\bar{\mathbf{k}}^{+} + \gamma\boldsymbol{dk}$ has the same properties as $\bar{\mathbf{k}}^{+}$.

Proposition 18 (Effect of a faction-mean-preserving spread of observable characteristics):† *Let USW(γ) and APW(γ) denote, respectively, equilibrium unbiased social and aggregate private welfare for the unique NMPE of the n player SQAG satisfying assumption A8, whose right-wing faction has the profile of observable characteristics* $\bar{\mathbf{k}}^{+} + \gamma \boldsymbol{dk}$.

i) *if* $\max(\bar{\mathbf{k}}^{+}) < 1/4n$, *then* $\left.\frac{dUSW(\gamma)}{d\gamma}\right|_{\gamma=0} < 0$ *and* $\left.\frac{dAPW(\gamma)}{d\gamma}\right|_{\gamma=0} < 0$

ii) *if* $\min(\bar{\mathbf{k}}^{+}) > 1/4n$, *then* $\left.\frac{dUSW(\gamma)}{d\gamma}\right|_{\gamma=0} > 0$

To obtain intuition for this surprising result, we return to Figure 3. Consider r with $k_r < 0$. Intuitively, the magnitude of $V\xi_r$ increases with the magnitude of r's involuntary distortion triangle. This triangle increases with the square of r's low threshhold type, $\underset{\sim}{\theta}_r$. Hence $V\xi_r$ is convex in r's threshhold type. On the other hand, in a symmetric game with a uniform distribution over types, r's threshhold type is a concave function of r's expected deviation from affine. The curvature of the convolution relating $V\xi_r$ to k_r depends on the balance between these two effects.

7. Single bounded games

In many applications, it is natural to assume that the announcement space is bounded at one end but not the other. The most obvious example is when announcements are restricted to be non-negative but there is no natural upper bound. (For example, agents might be reporting prices, interest rates or the variances of some privately observed statistic.) We refer to games satisfying this condition as *single-bounded aggregation games*. Naturally, the upper bound on actions in a single-bounded game should be infinite. However, to maintain consistency with the framework laid out in §3, we impose an artificial upper bound that will never bind. Since from (15), no player's equilibrium announcement will exceed $n\left(\max(\mathbf{k}) + \bar{\theta} - \underline{a}\right) + \underline{a}$, we impose in this section w.l.o.g:

Assumption A9: $\underline{a} = \underline{\theta} = 0$, and $\bar{a} = n\left(\max(\mathbf{k}) + \bar{\theta}\right)$.

A9 implies that the announcement space of a single-bounded game is inclusive, as well as:

$$\lambda_r \geq 0 \quad \Longrightarrow \quad E\xi_r(\lambda_r) = 0 \tag{37}$$

In a single-bounded game, a key role is played by the player h whose observable characteristic exceeds that of any other player. Note that since $\sum_i k_i = 0$, k_h is necessarily positive.

Proposition 19 (Single Bounded Games):† *Every single-bounded quadratic aggregation game satisfying A1-A6,A9 has a unique MPE* $\boldsymbol{\lambda}^*$ *in which* $\lambda_h^* \geq 0$ *and* $E\xi_h(\lambda_h^*) = 0$. *Moreover, for all* $r \neq h$, $\lambda_r^* \in int(\Lambda)$ *implies* $E\xi_r(\lambda_r^*) = n(k_h - k_r) > 0$.

Since $E\xi_h(\lambda_h^*) = 0$, every single-bounded game is anchored, with h as the anchor. While aggregation games satisfying assumption A9 look and feel quite different from the symmetric games studied in §6, the comparative statics properties we obtain in this section and in §6 are remarkably similar, at least for games in which the spread of $\mathbf{k}$ is small enough relative to n that an NMPE exists.[25] The similarity of the properties they exhibit is an indicator of the importance of the dominant role played by the anchor. We begin by characterizing the equilibrium of an arbitrary single-bounded game, then discuss its comparative statics properties. To avoid repetition, no formal results will be presented; we merely relate these properties to the corresponding properties derived in §6.

Since $E\xi_h = 0$, Prop. 7 implies that the equilibrium outcome implements h's CIPO outcome in expectation. Since $\underline{a} = \underline{\theta}$, $E\xi_r > 0$ for $r \neq h$ implies $\lambda_r < \underline{a} - \underline{\theta} = 0$. That is, every other player, even including one whose observable characteristics is very close to h's, will under-report to counteract h's extreme over-reporting. Indeed, from (15′) and Prop. 11,[26] $\sum_i \lambda_i = n(1-n)k_h < 0$; i.e., when $n > 2$, h's over-reporting is more than compensated by the sum of all other players' (unconstrained) under-reporting. Since player $r \neq h$ is constrained by the lower bound $\underline{a}$, his expected CIPO outcome differs from the expected equilibrium outcome. From Prop. 19, r's expected deviation from affine $E\xi_r = n(k_h - k_r)$, is greater the more different is r's characteristic from h's.

7.1. **Effects of changing the number of players.** The effects of increasing n in a single-bounded game are similar in most respects to the effects analyzed in §6.1. As in a symmetric game, r's expected deviation from affine is proportional to n—in this case, if $\boldsymbol{\lambda}^*$ is an equilibrium profile then $E\xi_r(\lambda_r^*) = n(k_h - k_r) > 0$—while r's expected CIPO deviation, $(k_h - k_r) > 0$, is independent of n. The expression for r's expected payoff is identical to the expression between the equality signs in (25′), except that the k_i's are replaced by $(k_h - k_i)$'s. The comparative statics of USW and expected payoffs w.r.t. n are comparable to those summarized in Prop. 14. The one striking difference between symmetric and single-bounded games concerns the strategic role played by

[25]It is straightforward to identify conditions analogous to assumption A7(iii) guaranteeing existence of an NMPE. To save space, we leave this as an exercise for the reader.

[26]Using (15′) then Prop. 11, and then assumption A1(i), we obtain:

$$nk_h = \sum_i \lambda_i + \sum_{i \neq h} E\xi_i(\lambda_i) = \sum_i \lambda_i + n \sum_{i \neq h}(k_h - k_i) = \sum_i \lambda_i + n^2 k_h.$$

the anchor player. A symmetric game is anchored by the middle player m, whose role is entirely passive: regardless of who else is playing the game, $\lambda^*_m = 0$. A single-bounded game is anchored by player h, whose strategy λ^*_h plays a pivotal equilibrating role. For $r \neq h$, r's expected CIPO deviation is positive and independent of n, in spite of the fact that as n increases, each new player contributes an additional downward bias to the mean report (i.e., $r \neq h \implies \lambda^*_r < 0$)! This balancing act is accomplished single-handedly by h, whose positive bias offsets the sum of all other players' negative biases. More precisely, from (15′), $\lambda_h \; = \; nk_h \; - \; \sum_{i \neq h}(E\xi_i(\lambda_i) + \lambda_i)$; since each term in the summation is negative, λ_h increases super-proportionally as n increases.

7.2. **Effects of changing the announcement bound.** Suppose the lower announcement bound, $\underline{a}$, decreases, holding $\underline{\theta}$ constant at zero, ensuring that the announcement space remains inclusive. The effects of this change are identical to those discussed in §6.2: each player's strategy adjusts to hold constant the first and second moments of his deviation from affine; the equilibrium outcome remains unchanged, as do all players' expected payoffs.

7.3. **Effects of increasing player heterogeneity.** Once again, the effects here are qualitatively similar to the effects described in §6.3-6.4. In the present context, we interpret an increase in heterogeneity as an increase in all components of the gap vector $\boldsymbol{\Delta k} = (k_h - k_i)_{i \neq h}$. Such a change unambiguously lowers all players' expected payoffs and USW. The proof closely parallels the proof of Prop. 17. Again, it is more interesting to consider the impact of a mean-preserving spread of $\boldsymbol{\Delta k}$. If we impose assumption A7 and let $[\underline{\theta}, \overline{\theta}] = [0, 1]$, the result we obtain is very similar to Prop. 18: if the largest element of $\boldsymbol{\Delta k}$ is less than $1/4n$, USW declines with a mean-preserving spread of $\boldsymbol{\Delta k}$; if the smallest element is greater than $1/4n$, USW increases.

8. Large Aggregation Games

The comparative statics results we present in §6 and §7 above apply to games in which the number of players is sufficiently small relative to the spread $||\mathbf{k}||_\infty$ of observable characteristics that an NMPE exists. Props. 2 and 3 reveal why for a given value of $||\mathbf{k}||_\infty$, there is an upper bound on how many players can participate beyond which some players' strategies become degenerate: as the population expands, the tug-of-war between players with different biases becomes so intense that more and more of them are driven to the boundaries of the strategy space, resulting in increasingly degenerate outcomes. Props. 20 and 21 below make this idea precise, first for bi-bounded games and then for single-bounded ones. In each case, we allow n to increase without bound, and demonstrate that in the limit, the outcome of the game is independent of players' realized signals.

The driving force behind these results is rational exaggeration. Each player in our model wants to distort the average signal that the center receives by an amount that is independent of n. But as n increases, a single individual's leverage over the average declines, so that more exaggeration is required in order to accomplish a given impact on the aggregate outcome. When the space of admissible reports is compact and n is sufficiently large, a right-winger, even if his type realization is close to the center, will be driven to the upper boundary of the admissible report space in a vain attempt to shift the mean announcement to the right. That is, compactness of the space of admissible announcements bounds the extent that a player can exaggerate: the best a right-winger can do is to select the highest admissible announcement, regardless of his type. Once this bound is reached, all connection between the player's private signal and his announcement is severed. As n gets larger, first extremists, then moderates, are pushed to this corner; increasingly, the boundary values of the announcement space dominate the determination of the mean signal, and the impact of private information shrinks to zero. This result contrasts sharply with the recurring theme in the information aggregation literature, which is that when the number of participants is very large, political institutions such as elections can effectively aggregate private information.

To formalize this argument we return from anchored games to the general specification laid out in §3. We consider an sequence of games with an increasingly large number of players, but with a fixed type-space $\Theta \subset \mathbb{R}$, type density h and announcement space, A. We also fix a set $K \subset \mathrm{int}(A)$ from which players' observable characteristics are drawn. Now for each n, let $\mathbf{k}^n = (k_r^n)_{r=1}^n \in K^n$ be a vector of observable characteristics satisfying assumption A1 and let Γ^n denote the n-player aggregation game satisfying A1-A6 defined by $\mathbf{k}^n$. Let ν^n denote the finite support measure on K defined by $\mathbf{k}^n$, i.e., for $k \in K$, $\nu^n(\{k\}) = \begin{cases} 1/n & \text{if } k = k_r^n, \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$. Let $\mathbf{s}^n$ be an MPE for Γ^n and let $t^n : \Theta^n \to \mathbb{R}$ denote the equilibrium outcome function, i.e., for $\boldsymbol{\theta} \in \Theta^n$, $t^n(\mu(\boldsymbol{\theta})) = t(\mathbf{s}^n(\boldsymbol{\theta}), \mathbf{k}^n)$. Passing to a subsequence if necessary, we can assume w.l.o.g. that the sequence $\{\nu^n\}$ converges weakly to a measure ν^* on K, and hence that $t^* = \lim_n t^n$ exists.

We impose the restriction that the $\mathbf{k}^n$'s do not "bunch up," i.e.,

Assumption A10: For any sequence (U^n) in K s.t. $\forall n$, diameter$(U^n) < 1/n$, $\lim_{n\to\infty} \nu^n(U^n) = 0$.

Assumption A10 implies that the limit measure ν^* is nonatomic. Although the results below hold more generally, we streamline the exposition by assuming that players' payoff functions are biased quadratic loss functions (see (13)) and that the mean of the signal distribution h is 0. Since by the Strong Law of Large Numbers, the mean of players' signals $\mu(\boldsymbol{\theta})$ converges almost surely to $E_\vartheta \vartheta = 0$ as n increases, the limit of player r's MPE payoffs is $-(k_r - t^*(0))^2$.

The striking property of large aggregation games is that asymptotically, there is no causal link between the mean of players' signal and the limit outcome of the game. Specifically, Prop. 20 establishes that as $n \to \infty$, the limit outcome $t^*(\cdot)$ is a constant function,[27] mapping all values of $\mu(\boldsymbol{\theta})$ to the same convex combination of the lower and upper boundaries, $\underline{a}$ and $\bar{a}$, of A.

Proposition 20 (Asymptotic information transmission):[†] *If the sequence of observable characteristics $\mathbf{k}^n$ satisfies assumption A10, then the limit outcome t^* of the sequence of games (Γ^n) is defined by $t^*(\cdot) = k^*$, where k^* is defined implicitly by the condition: $k^* = \bar{a} - \nu^*(\{k < k^*\})(\bar{a} - \underline{a})$.*

If the limit game is perfectly symmetric about zero—for example, is the limit of SQAG's—then $k^* = 0$ and $\nu^*(\{k < k^*\}) = 0.5$; in this case, the limit game will, by happenstance, effectively aggregate private information and the limit solution will maximize the limit of USW. In general, however, the solution will be suboptimal, to a degree that depends on the asymmetry of the distribution of observable characteristics and of the announcement bounds.

Prop. 20 is a straightforward consequence of parts iii) and iv) of Prop. 2, which reflect the fact that as n increases, players must exaggerate more and more, to exert the same degree of influence over the average outcome. But there are bounds on how much players can exaggerate, and once these bounds are attained, all connection between players' announcements and their signals is broken. It follows that as n increases, the fraction of players whose strategies convey any information at all about their signals shrinks to zero. Because the limit distribution over players' observable characteristics is non-atomic, the aggregation rule assigns vanishingly small weight to the information that these few players provide.

[27] As noted, in the limit, $\mu(\boldsymbol{\theta}) = 0$ with probability one. However, t^* is *defined* for all $\theta \in \Theta$.

We now consider single-bounded games. The only changes relative to the bi-bounded specification above are that we relax the restrictions that the set K is fixed and the mean signal is zero, imposing instead assumption A9 for each n, so that in effect, the upper bound on admissible announcements is removed. Prop. 20 was driven by the restriction that announcements were restricted to a fixed compact interval. Surprisingly, Prop. 21 delivers a similar result, even though announcements are bounded only from below. The difference between the two results is that in Prop. 20, the outcome depended on the distribution of players' biases. In Prop. 21, the player h with the largest positive bias dominates the game for each n (cf. Prop. 19), obtaining in the limit the highest possible payoff of zero, implemented by the limit outcome function $t(\cdot) = \lim_n k_h^n + E_\vartheta \vartheta$. As usual, we omit the proof because it is so similar to that of Prop. 20.

Proposition 21 (Limit of equilibria in single-bounded games): *Consider a sequence of games* (Γ^n) *satisfying assumption A9. If the sequence of observable characteristics* $\mathbf{k}^n$ *satisfies assumption A10, then the limit of the sequence of outcomes for* (Γ^n) *is* $t^*(\cdot) = \lim_n k_h^n + E_\vartheta \vartheta$.

9. Summary

This paper contributes to the literature on information aggregation. Two features that distinguish it from the mainstream of this literature is that players' reports are aggregated by averaging rather than majority rule, and their strategy set is an interval rather than a binary choice. In this context, the bounds on the strategy set play a critical role: if a group of players have distinct preferences, then all but at most one of them will be constrained by the bounds with positive probability. Our main general results are: if agents have identical preferences, information is perfectly transmitted, regardless of n; if there is any degree of preference heterogeneity, however, private information is entirely obliterated as n approaches infinity. For games with a small number of players, we establish a number of comparative statics results for a class of games with quadratic payoffs which we call anchored games: equilibrium outcomes and players' payoffs are independent of the size of the strategy set; as n increases, payoffs and social welfare tend to decline, but not necessarily monotonically; a mean-preserving increase in the heterogeneity of players' payoffs reduces payoffs and welfare, but if the player set is split into two symmetric factions, then an increase in the heterogeneity of each faction will under some conditions increase payoffs and welfare.

REFERENCES

Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information, *Econometrica* **69**(4): 861–890.

Austen-Smith, D. and Banks, J. (1996). Information aggregation, rationality, and the condorcet jury theorem, *American Political Science Review* **90**: 34–45.

Bernardo, J. M. and Smith, A. F. M. (2000). *Bayesian Theory*, John Wiley and Sons, Ltd, England.

British Banking Assoc (n.d.). Welcome to bbalibor, `http://www.bbalibor.com/bbalibor-explained/the-basics`.

Carson, R., Flores, N. and Meade, N. (2001). Contingent valuation: controversies and evidence, *Environmental and Resource Economics* **19**(2): 173–210.

Condorcet, M. (1785). *Essai sur l'application de l'analyse ai la probability des decisions rendues a la plurality des voix*.

Crawford, V. P. and Sobel, J. (1982). Strategic information transmission, *Econometrica* **50**: 1431–1451.

Dewan, T. and Shepsle, K. A. (2008). Review article: Recent economic perspectives on political economy, part i, *British Journal of Political Science* (38): 363–382.

Feddersen, T. and Pesendorfer, W. (1997). Voting behavior and information aggregation in elections with private information, *Econometrica* **65**(5): 1029–1058.

Gerling, K., Gruner, H. P., Kiel, A. and Schulte, E. (2005). Information acquisition and decision making in committees: a survey, *European Journal of Political Economy* pp. 563–597.

Gilligan, T. W. and Krehbiel, K. (1989). Asymmetric information and legislative rules with a heterogeneous committee, *American Journal of Political Science* **33**: 459–490.

Gross, D. (n.d.). The Shipping News: the best economic indicator you've never heard of, `http://www.slate.com/toolbar.aspx?action=print&id=2090303`.

Gruner, H. P. and Kiel, A. (2004). Collective decisions with interdependent valuations, *European Economic Review* **48**.

Hansen, S. W. (n.d.). Understanding The Baltic Dry Index, `http://www.learningmarkets.com/Stocks-Intermarket/20090304774/understanding-the-baltic-dry-index.html`.

Jickling, M. (2008). Regulation of Energy Derivatives, Congressional Research Service, Library of Congress.

Koontz, S. (1999). Accuracy of Fed Cattle Price Reporting: Is Mandatory Price Reporting Needed?, *Proceedings of the NCR-134 Conference on Applied Commodity Price Analysis, Forecasting, and Market Risk Management* pp. 53–68.

Krishna, V. and Morgan, J. (2001). A model of expertise, *Quarterly Journal of Economics* **116**: 747–775.

Leach, P. (2010). Hedging on Rate Futures, *Journal of Commerce* .

Lohmann, S. (1993). A signaling model of informative and manipulative political action, *The American Political Science Review* **87**(2): 319–333.

MacKenzie, D. (2008). What's in a number?, *London Review of Books* **30**(18): 11–12.

McLennan, A. (1998). Consequences of the condorcet jury theorem for beneficial information aggregation by rational agents, *American Political Science Review* (2): 413–418.

Milgrom, P. and Shannon, C. (1994). Monotone comparative statics, *Econometrica* **62**(1): 1089–1122.

Mollenkamp, C. and Whitehouse, M. (2008). Study Casts Doubt on Key Rate; WSJ Analysis Suggest Banks May Have Reported Flawed Interest Data for Libor, *Wall Street Journal* **1**.

Morgan, J. and Stocken, P. (2008). Information aggregation in polls, *American Economic Review* **98**(3).

Ortuno-Ortin, I. (1997). A spatial model of political competition and proportional representation, *Social Choice and Welfare* **14**(3): 427–438.

Ottaviani, M. and Squintani, F. (2006). Naive audience and communication bias, *International Journal of Game Theory* **35**(1): 129–150.

Piketty, T. (1999). The information-aggregation approach to political institutions, *European Economic Review* **43**: 791–800.

Razin, R. (2003). Signaling and election motivations in a voting model with common values and responsive candidates, *Econometrica* **71**(4): 1083–1119.

Snider, C. A. and Youle, T. (2010). Does the LIBOR Reflect Banks' Borrowing Costs?, *SSRN eLibrary* .

USGAO (2007). COMMODITY FUTURES TRADING COMMISSION: Trends in Energy Derivatives Markets Raise Questions about CFTC's Oversight, *Technical report*.

Wikinvest (n.d.). The Baltic Dry Index, http://www.wikinvest.com/index/Baltic_Dry_Index.

APPENDIX: PROOFS

Proof of Proposition 1: To prove the proposition we apply Theorems 1 and 2 of Athey (2001). The first of these theorems is used to establish existence for finite-action aggregation games. The second implies existence for general aggregation games. To apply Athey's first theorem, we define a *finite action aggregation game* to be one in which players are restricted to choose actions from a finite subset of A. In all other respects, finite action aggregation games are identical to (infinite action) aggregation games. We now check that u satisfies Athey's Assumption A1. Clearly, our types have joint density w.r.t. Lebesgue measure which is bounded and atomless. Moreover, the integrability condition in Athey's A1 is trivially satisfied since u is bounded. Moreover, inequality (7) implies that the SCC holds. Therefore, every finite action aggregation game has an MPE in which player r's equilibrium strategy s_r is nondecreasing. By Athey's Theorem 2, the restricted game has an MPE, call it $\mathbf{s}^*$. To show that $\mathbf{s}^*$ will also be an equilibrium for the original, unrestricted game, it suffices to show that for all r, all θ and all $a > \bar{a}$, $\frac{\partial U_r(a,\theta;\mathbf{s}^*_{-r})}{\partial a} < 0$. To establish this, note that $\mathbf{s}^*_{-r} \geq \underline{\mathbf{s}}_{-r} \geq 0$, so that since t is strictly increasing, $a > \bar{a}$ implies

$$U_r'(a,\theta;\mathbf{s}^*_{-r}) \quad < \quad U_r'(\bar{a},\theta;\mathbf{s}^*_{-r}) \quad \leq \quad U_r'(\bar{a},\theta;\underline{\mathbf{s}}_{-r}) \quad \leq \quad U_r'(\bar{a},\theta;0) \quad \leq \quad 0$$

Finally, to establish that s_r is strictly increasing and continuously differentiable on $(\underset{\sim}{\theta}_r(\mathbf{s}),\bar{\theta}]$, note that $U_r'(s_r(\cdot),\cdot;\mathbf{s}_{-r}) = 0$ on $(\underset{\sim}{\theta}_r(\mathbf{s}),\bar{\theta}]$. From (7), assumption A6 and the implicit function theorem, we have, for all $\theta \in (\underset{\sim}{\theta}_r(\mathbf{s}),\bar{\theta}]$, $\frac{ds_r(\theta)}{d\theta} = -\frac{\partial^2 U_r(s_r(\theta),\theta;\mathbf{s}_{-r})}{\partial a \partial \theta} \Big/ \frac{\partial^2 U_r(s_r(\theta),\theta;\mathbf{s}_{-r})}{\partial a^2} > 0$. ■

Proof of Proposition 2: Let $\mathbf{s}$ be an MPE and assume that $k_i - k_j > \varepsilon > 0$. Pick $\theta^* \in \Theta^* = \operatorname{argmin}(s_i - s_j)$ and let $\gamma = s_i(\theta^*) - s_j(\theta^*)$, so that $s_i(\cdot) - \gamma \geq s_j(\cdot)$. Thus, γ is the minimum amount by which $s_i(\cdot)$ exceeds $s_j(\cdot)$; we will establish $\gamma \geq 0$. Note first that

$$\begin{aligned} U_j'(s_j(\theta^*),\theta^*;\mathbf{s}_{-j}) \quad &\equiv \quad U_j'(s_i(\theta^*)-\gamma,\theta^*;\langle s_i,\mathbf{s}_{-i,j}\rangle) \quad = \quad U_j'(s_i(\theta^*),\theta^*;\langle s_i-\gamma,\mathbf{s}_{-i,j}\rangle) \\ &< \quad U_i'(s_i(\theta^*),\theta^*;\langle s_i-\gamma,\mathbf{s}_{-i,j}\rangle) \quad \leq \quad U_i'(s_i(\theta^*),\theta^*;\langle s_j,\mathbf{s}_{-i,j}\rangle) \end{aligned} \tag{38}$$

The first equality merely relabels some terms; the second equality hold because the outcome function satisfies condition (3). The strict inequality holds because by assumption A5(ii), $k_j < k_i$ implies that $U_j' < U_i'$. The weak inequality holds because U_i is concave w.r.t. $\mathbf{s}_{-i}$ (display (6)) and $s_i(\cdot) - \gamma \geq s_j(\cdot)$. It now follows from (38) that if $U_i'(s_i(\theta^*),\theta^*;\mathbf{s}_{-i}) \leq 0$, then $U_j'(s_j(\theta^*),\theta^*;\mathbf{s}_{-j}) < 0$, implying that $s_j(\theta^*) = \underset{\sim}{a}$, while if $U_i'(s_i(\theta^*),\theta^*;\mathbf{s}_{-i}) > 0$, then $s_i(\theta^*) = \bar{a}$. In either case, $s_i(\theta^*) - s_j(\theta^*) \geq 0$. Hence, by definition of θ^*,

$$0 \quad \leq \quad s_i(\theta^*) - s_j(\theta^*) \quad \leq \quad s_i(\cdot) - s_j(\cdot), \tag{39}$$

i.e., i's strategy is never lower than j's strategy. Thus $s_j(\theta) > \underset{\sim}{a}$ implies $s_i(\theta) > \underset{\sim}{a}$, implying in turn $\underset{\sim}{\theta}_i(\mathbf{s}) \geq \underset{\sim}{\theta}_j(\mathbf{s})$; and $s_i(\theta) < \bar{a}$ implies $s_j(\theta) < \bar{a}$, implying in turn $\tilde{\theta}_j(\mathbf{s}) \leq \tilde{\theta}_i(\mathbf{s})$, proving part i). To prove part ii), note that for $\theta \in \big(\underset{\sim}{\theta}_j(\mathbf{s}), \tilde{\theta}_i(\mathbf{s})\big)$,

$$\begin{aligned} U_i'(s_i(\theta),\theta;\langle s_j,\mathbf{s}_{-i,j}\rangle) \quad &= \quad 0 \quad = \quad U_j'(s_j(\theta),\theta;\langle s_i,\mathbf{s}_{-i,j}\rangle) \\ &\leq \quad U_j'(s_j(\theta),\theta;\langle s_j,\mathbf{s}_{-i,j}\rangle) \quad < \quad U_i'(s_j(\theta),\theta;\langle s_j,\mathbf{s}_{-i,j}\rangle) \end{aligned} \tag{40}$$

The equalities hold because neither i nor j is constrained at type θ. The weak inequality follows from property (6) since, from (39), $s_i \geq s_j$, and the strict inequality is implied by A5(ii). The inequality between the first and last expressions of (40), combined with (6), imply that $s_i(\theta) > s_j(\theta)$, proving part ii). To prove part iii), note first that since $\frac{\partial^2 u}{\partial\tau\partial k}, \frac{\partial^2 u}{\partial\tau\partial\mu(\boldsymbol{\theta})} > 0 > \frac{\partial^2 u}{\partial\mu(\mathbf{a})^2}$ (assumptions A5 and A6), since u is bounded and the domain of u is compact, there exists $\delta, \omega_\theta, \omega_a > 0$ such that $\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\tau\partial k} > 2\delta$, $\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\tau\partial\mu(\boldsymbol{\theta})} < \omega_\theta$ and $\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\mu(\mathbf{a})^2} \in (-\omega_a, 0)$, so that for all n, $\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\tau\partial\theta} = \frac{1}{n^2}\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\tau\partial\mu(\boldsymbol{\theta})} < \omega_\theta/n^2$ while $\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial a^2} = \frac{1}{n^2}\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\mu(\mathbf{a})^2} \in (-\omega_a/n^2, 0)$. Now fix $\hat{\theta} \in \big(\underset{\sim}{\theta}_j(\mathbf{s}), \tilde{\theta}_j(\mathbf{s})\big)$ so that j's first order condition is satisfied with equality at $\hat{\theta}$. From the strict inequality in (40), the lower bound on $\frac{\partial^2 u(\cdot,\cdot,\cdot)}{\partial\tau\partial k}$ and the fact that $(k_i - k_j > \varepsilon)$, we can infer that

$$U_i'(s_j(\hat{\theta}),\hat{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) \quad > \quad 2\varepsilon\delta/n. \tag{41}$$

Moreover, using the bounds just identified, we have that for $n > \frac{\max\{(\bar{a}-\underset{\sim}{a})\omega_a, (\bar{\theta}-\underline{\theta})\omega_\theta\}}{\varepsilon\delta}$,

$$\begin{aligned} U_i'(s_j(\hat{\theta}),\hat{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) - U_i'(\bar{a},\hat{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) \quad &\equiv \quad -\int_{s_j(\hat{\theta})}^{\bar{a}} \frac{dU_i'(\alpha,\hat{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle)}{da} d\alpha \\ &\leq \quad \frac{\omega_a(\bar{a}-\underset{\sim}{a})}{n^2} \quad < \quad \varepsilon\delta/n \end{aligned} \tag{42}$$

$$\begin{aligned} \text{while} \qquad U_i'(\bar{a},\hat{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) - U_i'(\bar{a},\underline{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) \quad &\equiv \quad \int_{\underline{\theta}}^{\hat{\theta}} \frac{dU_i'(\bar{a},\vartheta;\langle s_j,\mathbf{s}_{-i,j}\rangle)}{d\theta} d\vartheta \\ &\leq \quad \frac{\omega_\theta(\bar{\theta}-\underline{\theta})}{n^2} \quad < \quad \varepsilon\delta/n \end{aligned} \tag{43}$$

Inequalities (42) and (43) together imply that $U_i'(s_j(\hat{\theta}),\hat{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) - U_i'(\bar{a},\underline{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) < 2\varepsilon\delta/n$, which, together with (41), implies that $U_i'(\bar{a},\underline{\theta};\langle s_j,\mathbf{s}_{-i,j}\rangle) > 0$. It now follows from (8) and monotonicity of $s_i(\cdot)$ that $\bar{a} = s_i(\underline{\theta}) \leq s_i(\cdot)$, establishing part iii). The proof of iv) is parallel. ■

Proof of Proposition 5: We first assume that $\mathbf{s}$ is admissible and unit affine but not ZSUA, i.e., that there exists $\boldsymbol{\lambda} \in \Lambda^2$ such that $s_r(\cdot) = \iota(\cdot) + \lambda_r$, with $\sum_{r=i}^{j} \lambda_r \neq 0$. Assume w.l.o.g. that $\lambda_i \geq 0$ and that $|\underset{\sim}{a} - \bar{\theta}| > |\bar{a} - \underline{\theta}|$, implying that $-\lambda_i \in \Lambda$. Fix θ_j arbitrarily:

$$U_j(s_j(\theta_j),\theta_j;s_i) \quad = \quad \int_\Theta u(t(\vartheta_i+\lambda_i,\theta_j+\lambda_j),(\vartheta_i,\theta_j),\bar{k})dh(\vartheta_i)$$

which, since t is CISE

$$< \quad \int_{\Theta} u(t(\vartheta_i,\theta_j),(\vartheta_i,\theta_j),\bar{k})dh(\vartheta_i) \quad = \quad U_j(\theta_j-\lambda_i,\theta_j;s_i)$$

That is, $s_j(\cdot)$ is not a best response against s_i so that $\mathbf{s}$ is not an equilibrium profile.
Now assume that $\mathbf{s}$ is continuous but not unit affine. (From Prop. 1, we do not need to consider discontinuous strategies.) Note also that for f and g continuous, $f \gneqq g$ implies that f strictly exceeds g with positive probability. W.l.o.g., assume that there exists $\lambda > 0$ such that $s_j(\cdot) \gneqq \iota(\cdot)-\lambda$, with $s_j(\underline{\theta}_j)=\underline{\theta}_j-\lambda$. We now show that if s_i is a best response to s_j, then $(s_i(\cdot)-\iota(\cdot))<\lambda$. Consider s_i such that $s_i(\bar{\theta}_i)\geq\bar{\theta}_i+\lambda$, for some $\bar{\theta}_i$, so that $s_i(\bar{\theta}_i)+s_j(\cdot)\geq\bar{\theta}_i+\lambda+s_j(\cdot)\gneqq\bar{\theta}_i+\iota(\cdot)$. Fact (5) on p. 13 now implies $t(s_i(\bar{\theta}_i),s_j(\cdot))\geq t(\bar{\theta}_i+\lambda,s_j(\cdot))\gneqq t(\bar{\theta}_i,\iota(\cdot))$. Since U_i is concave in t and, for all θ_j, $u(\cdot,(\bar{\theta}_i,\theta_j),\bar{k})$ is maximized at $t(\bar{\theta}_i,\theta_j)=\mu\big(\bar{\theta}_i,\theta_j\big)$:

$$\begin{aligned}
U_i'(s_i(\bar{\theta}_i),\bar{\theta}_i;s_j) &= \int_{\Theta}\frac{du}{da}(t(s_i(\bar{\theta}_i),s_j(\vartheta_j)),(\bar{\theta}_i,\vartheta_j),\bar{k})dh(\vartheta_i) \;\leq\; \int_{\Theta}\frac{du}{da}(t(\bar{\theta}_i+\lambda,s_j(\vartheta_j)),(\bar{\theta}_i,\vartheta_j),\bar{k})dh(\vartheta_i)\\
&< \int_{\Theta}\frac{du}{da}(t(\bar{\theta}_i,\vartheta_j),(\bar{\theta}_i,\vartheta_j),\bar{k})dh(\vartheta_i) \;=\; 0
\end{aligned}$$

This establishes that if s_i is a best response to s_j, then $(s_i(\cdot)-\iota(\cdot))<\lambda$. But in this case, $s_j(\bar{\theta}_j)+s_i(\cdot)<\bar{\theta}_j+\iota(\cdot)$, implying that $t(s_j(\bar{\theta}_j),s_i(\cdot)) < t(\bar{\theta}_j,\iota(\cdot))$, so that $U_j'(s_j(\bar{\theta}_j),\bar{\theta}_j;s_i)>0$. Therefore, $s_j(\cdot)$ is not a best response for j against $s_i(\cdot)$ at $\bar{\theta}_j$. ■

Proof of Proposition 6: We will prove uniqueness only for non-degenerate equilibrium profiles. Uniqueness for other profiles is ensured by restriction (10), but we omit the details. Let $\boldsymbol{\lambda}^*$ be a NMPE for the aggregation game, and let $\boldsymbol{\lambda}$ be any other profile of strategies such that for some j, $\lambda_j\neq\lambda_j^*$. We will show that if $\boldsymbol{\lambda}$ satisfies the necessary condition (22), then it fails the other necessary condition (15′). Suppose w.l.o.g. that $\lambda_j>\lambda_j^*$. From (21), $E\xi_j(\lambda_j)<E\xi_j(\lambda_j^*)$. For all $r\neq j$, (22) implies that $E\xi_r(\lambda_r)<E\xi_r(\lambda_r^*)$, and (21) in turn implies that $\lambda_r>\lambda_r^*$. To establish that $\boldsymbol{\lambda}$ cannot satisfy (15′), it suffices to show that

$$\left(\sum_i\lambda_i + \sum_{i\neq j}E\xi_i(\lambda_i)\right) \quad > \quad \left(\sum_i\lambda_i^* + \sum_{i\neq j}E\xi_i(\lambda_i^*)\right) \quad = \quad nk_j$$

or, equivalently

$$\lambda_j - \lambda_j^* + \sum_{i\neq j}(\lambda_i - \lambda_i^*) \quad > \quad \sum_{i\neq j}(E\xi_i(\lambda_i^*) - E\xi_i(\lambda_i))$$

This last inequality is indeed satisfied, since by assumption $\lambda_j>\lambda_j^*$ while (21) implies that for all $i\neq j$, $\lambda_i-\lambda_i^*>E\xi_i(\lambda_i^*)-E\xi_i(\lambda_i)$. ■

Proof of Proposition 7: From (17), $E_{\boldsymbol{\vartheta}}\big(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta})\big)$ equals $\mu(\boldsymbol{\lambda}^*)+\mu(E\boldsymbol{\xi})$, which, from (15′), equals $k_r+E\xi_r/n$. Hence, from (23), $E_{\boldsymbol{\vartheta}}\big(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\hat{t}(\boldsymbol{\vartheta},k_r)\big)=E\xi_r/n$. ■

Proof of Proposition 8: Rearranging (15), we obtain the *interim expected equilibrium outcome*

$$E_{\boldsymbol{\vartheta}_{-r}}(\mu(\mathbf{s})|\theta_r) \quad = \quad \Big(\min\{\bar{a},\max\{\theta_r+\lambda_r,\underline{a}\}\} \quad + \quad nk_r+\sum_{i\neq r}E_{\boldsymbol{\vartheta}}\vartheta_i-\lambda_r\Big)/n$$

It follows that for (r,θ_r), the interim expected equilibrium and CIPO outcomes will coincide iff $\min\{\bar{a},\max\{\theta_r+\lambda_r,\underline{a}\}\}=\theta_r+\lambda_r$, i.e., (r,θ_r), is not constrained by the announcement bounds. ■

Proof of Proposition 9: Let $\xi_r^*=\xi_r(\lambda_r^*)$. Expanding the left hand side of (25), we obtain

$$\begin{aligned}
E_{\boldsymbol{\vartheta}}(\mu(\boldsymbol{\vartheta})+k_r-\mu(\mathbf{s}^*(\boldsymbol{\vartheta})))^2 \quad &= \quad E_{\boldsymbol{\vartheta}}(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta})-k_r)^2 \\
&= \quad E_{\boldsymbol{\vartheta}}(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta}))^2\;-\;2k_rE_{\boldsymbol{\vartheta}}(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta}))\;+\;k_r^2 \\
&= \quad E_{\boldsymbol{\vartheta}}(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta}))^2\;-\;2k_r\big(\mu\left(E\boldsymbol{\xi}^*\right)+\mu(\boldsymbol{\lambda}^*)\big)\;+\;k_r^2 \qquad (44)
\end{aligned}$$

The last equality follows from (20). Expanding the first term on the right hand side of (44),

$$\begin{aligned}
E_{\boldsymbol{\vartheta}}\left(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta})\right)^2 \quad &= \quad E_{\boldsymbol{\vartheta}}\Big(\mu\big(\underbrace{\mathbf{s}^*(\boldsymbol{\vartheta})-(\boldsymbol{\vartheta}+\boldsymbol{\lambda}^*)}_{\boldsymbol{\xi}^*}\big)+\mu(\boldsymbol{\lambda}^*)\Big)^2 \\
= \quad E_{\boldsymbol{\vartheta}}\big(\mu(\mathbf{s}^*(\boldsymbol{\vartheta})-(\boldsymbol{\vartheta}+\boldsymbol{\lambda}^*))\big)^2\;&+\;2\mu(\boldsymbol{\lambda}^*)\mu\left(E\boldsymbol{\xi}^*\right)\;+\;\mu(\boldsymbol{\lambda}^*)^2 \qquad (45)
\end{aligned}$$

The first equality merely adds and subtracts $\mu(\boldsymbol{\lambda}^*)$ and rearranges terms; the second averages both sides of the identity in (17). Now expand the first term in (45) to obtain

$$\begin{aligned}
E_{\boldsymbol{\vartheta}}\big(\mu(\mathbf{s}^*(\boldsymbol{\vartheta})-(\boldsymbol{\vartheta}+\boldsymbol{\lambda}^*))\big)^2 \quad &= \quad \Big(\sum_i E_{\vartheta_i}\left(s_i^*(\vartheta_i)-(\vartheta_i+\lambda_i^*)\right)^2\;+\;\sum\sum\nolimits_{i\neq j}E\xi_i^*E\xi_j^*\Big)/n^2 \\
= \quad \big(\sum_i V\xi_i^*\;+\;\sum_i(E\xi_i^*)^2+\sum\sum\nolimits_{i\neq j}E\xi_i^*E\xi_j^*\big)/n^2 \quad &= \quad \big(\sum_i V\xi_i^*\;+\;\big[\sum_i E\xi_i^*\big]^2\big)/n^2 \\
= \quad \mu\left(\mathbf{V}\boldsymbol{\xi}^*\right)/n\;+\;\left(\mu\left(E\boldsymbol{\xi}^*\right)\right)^2 & \qquad (46)
\end{aligned}$$

The first equality is obtained by expanding $\mu(\boldsymbol{\vartheta}+\boldsymbol{\lambda}^*-\mathbf{s}^*(\boldsymbol{\vartheta}))$, the second from the relationship $E(X^2)=Var(X)+(EX)^2$ for a random variable X. Now, substituting (46) back into (45)

$$E_{\boldsymbol{\vartheta}}\left(\mu(\mathbf{s}^*(\boldsymbol{\vartheta}))-\mu(\boldsymbol{\vartheta})\right)^2 \quad = \quad \mu\left(\mathbf{V}\boldsymbol{\xi}^*\right)/n\;+\;\left(\mu\left(E\boldsymbol{\xi}^*\right)+\mu(\boldsymbol{\lambda}^*)\right)^2 \qquad (47)$$

Finally, substitute (47) back into (44) to obtain

$$E_{\boldsymbol{\vartheta}}(\mu(\boldsymbol{\vartheta})+k_r-\mu(\mathbf{s}^*(\boldsymbol{\vartheta})))^2\;=\;\mu\left(\mathbf{V}\boldsymbol{\xi}^*\right)/n\;+\;\left(\mu\left(E\boldsymbol{\xi}^*\right)+\mu(\boldsymbol{\lambda}^*)-k_r\right)^2\;=\;\mu\left(\mathbf{V}\boldsymbol{\xi}^*\right)/n\;+\;(E\xi_r^*/n)^2$$

The last equality is obtained by adding $E\xi_r^*/n$ to both sides of (15′) and substituting for k_r. ■

Proof of Proposition 12: We first show that under A7, for the variance of r's deviation from affine is $V\xi_r(\lambda_r)=E\xi_r^2\left(\sqrt{\frac{8}{9h|E\xi_r|}}\;-\;1\right)$. To see this, assuming w.l.o.g. that $\lambda_r>0$, and using the fact that $\lambda_r \in \mathrm{int}(\Lambda)$, we have

$$V\xi_r(\lambda_r) \quad = \quad -E\xi_r^2 \quad + \quad \int_{\bar{a}-\lambda_r}^{\bar{\theta}} (\vartheta_r + \lambda_r - \bar{a})^2\, dh(\vartheta_r) \;=\; h/3\left(\lambda_r + \bar{\theta} - \bar{a}\right)^3 - E\xi_r^2 \tag{48}$$

which, from (29),

$$= \quad \frac{h}{3}\left(\frac{2}{h}E\xi_r\right)^{3/2} - E\xi_r^2 \quad = \quad \sqrt{\frac{8|E\xi_r|^3}{9h}} - E\xi_r^2 \quad = \quad E\xi_r^2\left(\sqrt{\frac{8}{9h|E\xi_r|}} - 1\right).$$

Prop. 9 now implies that r's equilibrium expected payoff is $-\left\{\sum_i \left[E\xi_i^2\left(\sqrt{\frac{8}{9h|E\xi_i|}} - 1\right)\right] + (E\xi_r)^2\right\}/n^2$. Equation (30) then follows from Prop. 11. The inequality follows since $|k_j - k_i| \le 2||\mathbf{k}||_\infty < (\bar{\theta} - \underline{\theta})/2n = 1/(2hn)$. ■

Proof of Proposition 13: The existence of a unique MPE was established in Prop. 6. Nondegeneracy is implied by assumption A7(iii) (see pp. 28-29). Consider $\boldsymbol{\lambda}^*$ such that $E\xi_r(\lambda_r^*) = -nk_r$ for all r with $\lambda_r \in \mathrm{int}(\Lambda)$ and parts i) and ii) of the proposition are satisfied. Our symmetry conditions ensure that such a vector exists, i.e., that if $\bar{r}$ and $\underline{r}$ are matched players, if $\lambda_{\underline{r}}^* = -\lambda_{\bar{r}}^*$, and $E\xi_{\bar{r}}(\lambda_{\bar{r}}^*) = -nk_{\bar{r}}$, it follows from symmetry, (17) and (18) that $E\xi_{\underline{r}}(\lambda_{\underline{r}}^*) = -nk_{\underline{r}}$. With the restrictions in (16), we only need to verify that (15′) is satisfied by $\boldsymbol{\lambda}^*$. Since $\sum_i k_i = 0$ (assumption A1), we have

$$-nk_r \quad = \quad \sum_{i \neq r} nk_i \quad = \quad -\sum_{i \neq r} E\xi_i(\lambda_i^*) \tag{49}$$

Moreover, from parts i) and ii) of the proposition, $\sum_i \lambda_i^* = 0$. Substituting this property and (49) into the right hand side of (15′), we obtain

$$\sum_i \lambda_i^* + \sum_{i \neq r} E\xi_i(\lambda_i^*) \quad = \quad nk_r,$$

verifying that (15′) is indeed satisfied. ■

Proof of Proposition 15: From Prop. 13, we have

$$E\xi_r \quad \equiv \quad \int_{\underline{\theta}}^{\underset{\sim}{\theta}_r} (-\bar{a} - (\theta_r + \lambda_r))dH(\theta_r) + \int_{\tilde{\theta}_r}^{\bar{\theta}} (\bar{a} - (\theta_r + \lambda_r))dH(\theta_r) \quad \equiv \quad -nk_r, \tag{50}$$

where in the first integration we substituted in $\underline{a} = -\bar{a}$. Totally differentiating both sides with respect to $\bar{a}$ and λ_r and noting that $\underset{\sim}{\theta}_r = \underline{a} - \lambda_r = -\bar{a} - \lambda_r$ and $\tilde{\theta}_r = \bar{a} - \lambda_r$, we obtain

$$\left[H(\underset{\sim}{\theta}_r) - (1 - H(\tilde{\theta}_r))\right] + \left[(H(\underset{\sim}{\theta}_r) + (1 - H(\tilde{\theta}_r))\right]\frac{d\lambda_r}{d\bar{a}} \quad = \quad 0.$$

Hence $\frac{d\lambda_r}{da} = \frac{(1-H(\tilde{\theta}_r)) - H(\underset{\sim}{\theta}_r)}{H(\underset{\sim}{\theta}_r) + (1-H(\tilde{\theta}_r))}$. When r is bi-constrained, $H(\underset{\sim}{\theta}_r)$ and $H(\tilde{\theta}_r)$ are both nonzero, so that $\frac{d\lambda_r}{da} \in (0,1)$. When r is up-constrained (resp. down-constrained), $H(\underset{\sim}{\theta}_r) = 0$ (resp. $H(\tilde{\theta}_r) = 1$), so that $\frac{d\lambda_r}{da}$ reduces to 1 (resp. -1). If r is the middle player, $\lambda_r = 0$ and, since everything is symmetric, $H(\underset{\sim}{\theta}_r) = 1 - H(\tilde{\theta}_r)$ so that $\frac{d\lambda_r}{da} = 0$. ■

Proof of Proposition 16: Since part i) of the proposition follows immediately from the discussion below Prop. 15, we need only prove in detail part ii). Suppose there is a player i whose strategy is bi-constrained. (If i is not the middle player, his matched player is also bi-constrained.) We will show that as $\overline{a}$ increases by da, the variance term $V\xi_i$ decreases, which, from (25′) induces the same increase of $-\frac{1}{n^2}\frac{dV\xi_i}{da}da$ in the expected payoff of each player. Let the distribution function of player i's deviation from affine, ξ_i, be denoted as $F_i(\cdot)$. Obviously $F_i(\cdot)$ is derived from the distribution function of θ, $H(\cdot)$, as well as from i's strategy and the announcement bounds. The random variable ξ_i can be considered as a function of random variable θ_i:

$$\xi_i \quad = \quad \begin{cases} \underline{a} - (\theta_i + \lambda_i) \quad = \quad \underset{\sim}{\theta}_i - \theta_i & \text{if } \theta_i \leq \underset{\sim}{\theta}_i \\ 0 & \text{if } \underset{\sim}{\theta}_i < \theta_i \leq \tilde{\theta}_i \\ \overline{a} - (\theta_i + \lambda_i) \quad = \quad \tilde{\theta}_i - \theta_i & \text{if } \theta_i > \underset{\sim}{\theta}_i \end{cases} \tag{51}$$

Given that θ_i is distributed according to $H(\cdot)$, the distribution function $F_i(\cdot)$ of ξ_i can be derived by combining $H(\cdot)$ and (51). Specifically, the support of F_i is $[\tilde{\theta}_i - \overline{\theta}, \underset{\sim}{\theta}_i - \underline{\theta}]$; the fact that i is bi-constrained implies that $\tilde{\theta}_i - \overline{\theta} < 0$ and $\underset{\sim}{\theta}_i - \underline{\theta} > 0$; The values of F_i are given by

$$F_i(x) \quad = \quad \begin{cases} \text{Prob}(\tilde{\theta}_i - \theta_i \leq x) = 1 - H(\tilde{\theta}_i - x) & x \in [\tilde{\theta}_i - \overline{\theta}, 0) \\ \text{Prob}(\theta_i \geq \tilde{\theta}_i) = 1 - H(\tilde{\theta}_i) & \text{if } x = 0 \\ \text{Prob}(\underset{\sim}{\theta}_i - \theta_i \leq x) = 1 - H(\underset{\sim}{\theta}_i - x) & x \in (0, \underset{\sim}{\theta}_i - \underline{\theta}]. \end{cases} \tag{52}$$

Note in particular that $F_i(\cdot)$ jumps up at $x = 0$ from $1 - H(\tilde{\theta}_i)$ to $1 - H(\underset{\sim}{\theta}_i)$. To derive the variance $V\xi_i$, note first that since i is bi-constrained, $\lambda_i \in \text{int}(\Lambda)$. We can therefore invoke Prop. 13 to obtain:

$$-nk_i \quad \equiv \quad E(\xi_i) \quad = \quad \int_{\tilde{\theta}_i - \overline{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} \xi_i dF_i(\xi_i) \quad = \quad \underset{\sim}{\theta}_i - \underline{\theta} - \int_{\tilde{\theta}_i - \overline{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} F_i(\xi_i) d\xi_i,$$

where the last equality is obtained after integrating by parts. Thus,

$$\int_{\tilde{\theta}_i - \overline{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} F_i(\xi_i) d\xi_i \quad = \quad \underset{\sim}{\theta}_i - \underline{\theta} + nk_i. \tag{53}$$

The variance of ξ_i can now be written as

$$V\xi_i \; = \; \int_{\tilde{\theta}_i - \overline{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} (\xi_i - E(\xi_i))^2 dF_i(\xi_i) \; = \; (\underset{\sim}{\theta}_i - \underline{\theta} - E(\xi_i))^2 \quad - \quad \int_{\tilde{\theta}_i - \overline{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} F_i(\xi_i) 2(\xi_i - E(\xi_i)) d\xi_i$$

$$
\begin{aligned}
&= (\underset{\sim}{\theta}_i - \underline{\theta} + nk_i)^2 \quad - \quad 2nk_i \int_{\tilde{\theta}_i - \bar{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} F_i(\xi_i)d\xi_i \quad - \quad 2\int_{\tilde{\theta}_i - \bar{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} F_i(\xi_i)\xi_i d\xi_i \\
&= (\underset{\sim}{\theta}_i - \underline{\theta})^2 + 2(\underset{\sim}{\theta}_i - \underline{\theta})nk_i + (nk_i)^2 \quad - \quad 2nk_i(\underset{\sim}{\theta}_i - \underline{\theta} + nk_i) \quad - \quad 2\int_{\tilde{\theta}_i - \bar{\theta}}^{\underset{\sim}{\theta}_i - \underline{\theta}} F_i(\xi_i)\xi_i d\xi_i \\
&= (\underset{\sim}{\theta}_i - \underline{\theta})^2 - (nk_i)^2 \; - \; 2\left[\int_{\tilde{\theta}_i - \bar{\theta}}^{0} (1 - H(\tilde{\theta}_i - \xi_i))\xi_i d\xi_i \; + \; \int_{0}^{\underset{\sim}{\theta}_i - \underline{\theta}} (1 - H(\underset{\sim}{\theta}_i - \xi_i))\xi_i d\xi_i\right], \qquad (54)
\end{aligned}
$$

where the second equality follows from integration by parts, the third from $E(\underline{\xi}_i) = -nk_i$, the fourth from (53) and the fifth from (52). Now, differentiating (54) with respect to $\bar{a}$ and noting that $\underset{\sim}{\theta}_i = \underset{\sim}{a} - \lambda_i = -\bar{a} - \lambda_i$ and $\tilde{\theta}_i = \bar{a} - \lambda_i$, we obtain

$$
\begin{aligned}
\frac{dV\xi_i}{d\bar{a}} \quad &= \quad 2\left\{\left[1 - \frac{d\lambda_i}{d\bar{a}}\right]\int_{\tilde{\theta}_i - \bar{\theta}}^{0} h(\tilde{\theta}_i - \xi_i)\xi_i d\xi_i \quad - \quad \left[1 + \frac{d\lambda_i}{d\bar{a}}\right]\int_{0}^{\underset{\sim}{\theta}_i - \underline{\theta}} h(\underset{\sim}{\theta}_i - \xi_i)\xi_i d\xi_i\right\} \\
&= \quad \frac{4}{H(\underset{\sim}{\theta}_i) + (1 - H(\tilde{\theta}_i))} \times \qquad (34') \\
&\qquad \left\{H(\underset{\sim}{\theta}_i)\int_{\bar{\theta}}^{\tilde{\theta}_i} (\theta_i - \tilde{\theta}_i)dH(\theta_i) \; - \; (1 - H(\tilde{\theta}_i))\int_{\underset{\sim}{\theta}_i}^{\underline{\theta}} (\theta_i - \underset{\sim}{\theta}_i)dH(\theta_i)\right\} \; < \; 0
\end{aligned}
$$

The first inequality holds because $H(\underline{\theta}) = 0$ and $H(\bar{\theta}) = 1$, while $\frac{d\underset{\sim}{\theta}_i}{d\bar{a}} \equiv \frac{d(\underset{\sim}{a} - \lambda_i)}{d\bar{a}} = -(1 + \frac{d\lambda}{d\bar{a}})$ and $\frac{d\tilde{\theta}_i}{d\bar{a}} \equiv \frac{d(\bar{a} - \lambda_i)}{d\bar{a}} = (1 - \frac{d\lambda}{d\bar{a}})$. The second equality is obtained by substituting in the value of $d\lambda_i / d\underset{\sim}{a}$ using (33), changing the variables of integration from ξ_i to $\theta_i = \tilde{\theta}_i - \xi_i$ and to $\theta_i = \underset{\sim}{\theta}_i - \xi_i$ in the two integrations respectively. The term in curly brackets is negative because $\underline{\theta} < \underset{\sim}{\theta}$ while $\bar{\theta} > \tilde{\theta}$. ■

Proof of Proposition 17: By symmetry, we can, w.l.o.g., assume that $k_r > 0$. Similar to the procedures used to derive (34′), we differentiate the expression for $V\xi_r$ in (54) w.r.t. λ_r, to obtain

$$
\frac{\partial V\xi_r}{\partial \lambda_r} = 2\int_{\underline{\theta}}^{\underset{\sim}{\theta}_r} (\theta_r - \underset{\sim}{\theta}_r)dH(\theta_r) + 2\int_{\tilde{\theta}_r}^{\bar{\theta}} (\theta_r - \tilde{\theta}_r)dH(\theta_r) = -2E\xi_r = 2nk_r, \qquad (55)
$$

where the last equality follows from Prop. 13. Note that if r is up-constrained, the first term in expression (55) is zero. Since $\frac{dV\xi_r}{dk_r} = \frac{\partial V\xi_r}{\partial k_r} + \frac{\partial V\xi_r}{\partial \lambda_r}\frac{d\lambda_r}{dk_r}$, the Proposition is obtained by taking the derivative of (54) with respect to k_r and combining (35) with (55). ■

Proof of Proposition 18: Let I^+ denote the members of the right-wing faction and let I^+_- denote the moderate members of this faction. Pick $r \in I^+$. Let $\xi_r(\gamma)$ denote r's deviation from affine in the equilibrium associated with the parameter γ. Since r is up-bounded, we have

$$
n\bar{k}_r^+ \quad = \quad -E\xi_r(0) \quad = \quad \int_{\tilde{\theta}_r}^{\bar{\theta}} (\theta_r - \tilde{\theta}_r)dH(\theta_r) \quad = \quad 0.5\int_{\tilde{\theta}_r}^{1} (\theta_r - \tilde{\theta}_r)d\theta \quad = \quad (1 - \tilde{\theta}_r)^2/4
$$

The first equality follows from Prop. 13 and the third from assumption A8(i). Hence $\tilde{\theta}_r = 1 - 2\sqrt{n\bar{k}_r^+}$. Moreover, $H(\tilde{\theta}_r) = 0.5\int_{-1}^{\tilde{\theta}_r} d\theta = \frac{1+\tilde{\theta}_r}{2}$. Now from (36)

$$\begin{aligned} \left.\frac{dV\xi_r}{dk_r}\right|_{\gamma=0} &= 2n^2\bar{k}_r^+\left(\frac{H(\tilde{\theta}_r)}{1-H(\tilde{\theta}_r)}\right) = 2n^2\bar{k}_r^+\left(\frac{1+\tilde{\theta}_r}{1-\tilde{\theta}_r}\right) = 2n^2\bar{k}_r^+\left(\frac{1-\sqrt{n\bar{k}_r^+}}{\sqrt{n\bar{k}_r^+}}\right) \\ &= 2n(\sqrt{n\bar{k}_r^+} - n\bar{k}_r^+) \end{aligned}$$

Hence $\left.\frac{d^2V\xi_r}{dk_r{}^2}\right|_{\gamma=0} = n\left(\sqrt{n}/\sqrt{\bar{k}_r^+} - 2n\right) \lesseqgtr 0$ as $\bar{k}_r^+ \gtreqless 1/4n$. That is, for $k' > k$, $\frac{dV\xi_r(k')}{dk} > \frac{dV\xi_r(k)}{dk}$ if $k' < 1/4n$ and $\frac{dV\xi_r(k')}{dk} < \frac{dV\xi_r(k)}{dk}$ if $k > 1/4n$. From Prop. 10, Prop. 13 and symmetry, $\text{USW} = -2\sum_{i\in I^+} V\xi_i(\gamma)$, so that

$$\left.\frac{d\text{USW}}{d\gamma}\right|_{\gamma=0} = -2\sum_{i\in I^+}\left.\frac{dV\xi_i(\gamma)}{d\gamma}\right|_{\gamma=0} = -2\sum_{i\in I^+_-}\alpha_i\left(\left.\frac{dV\xi_{i+n/4}(\gamma)}{dk_{i+n/4}}\right|_{\gamma=0} - \left.\frac{dV\xi_i(\gamma)}{dk_i}\right|_{\gamma=0}\right)$$

Since $\bar{k}_{i+n/4}^+ > \bar{k}_i^+$, $\left.\frac{d\text{USW}}{d\gamma}\right|_{\gamma=0} > 0$ if $\min(\bar{\mathbf{k}}^+) > 1/4n$ and $\left.\frac{d\text{USW}}{d\gamma}\right|_{\gamma=0} < 0$ if $\max(\bar{\mathbf{k}}^+) < 1/4n$. ■

Proof of Proposition 19: We first establish $\lambda_h^* > 0$, so that, from (37), $E\xi_h(\lambda_h^*) = 0$ and thus h is the anchor of the game. Suppose instead that $\lambda_h^* \leq 0$ and $\lambda_h^* \in \text{int}(\Lambda)$. (We can easily rule out the situation when $\lambda_h^* = \min(\Lambda) = \underline{a} - \bar{\theta}$; we omit the details.) Since $k_h > k_r\ \forall r \neq h$, (22) implies that $E\xi_h(\lambda_h^*) < E\xi_r(\lambda_r^*)$ and thus $\lambda_r^* < \lambda_h^* \leq 0$. Since $E\xi_r(\lambda_r) = 0$ when $\lambda_r = 0$, (21) and $\lambda_r^* < 0$ imply

$$\lambda_r^* + E\xi_r(\lambda_r^*) < 0. \tag{56}$$

From (15′) and $\lambda_h^* \in \text{int}(\Lambda)$, $\lambda_h^* = nk_h - \sum_{r\neq h}(\lambda_r^* + E\xi_r(\lambda_r^*)) > 0$, where the inequality is due to $k_h > 0$ and (56). This contradicts our supposition that $\lambda_h^* \leq 0$. Property (37) now ensures that $E\xi_r(\lambda_r) = 0$, so that single-bounded aggregation games are anchored with anchor h. The second part of the proposition now follows from Prop. 11. ■

Proof of Proposition 20: Let $\mathbf{s}^n$ denote the MPE of the n'th game and let $\bar{U}^n = \{\kappa \in K : \exists r \text{ s.t. } k_r^n = \kappa \text{ and } s_r^n(\cdot) = \bar{a}\}$. Define $\underline{U}^n$ analogously, with $\underline{a}$ replacing $\bar{a}$. Finally Let $U^n = \{\kappa \in K : \exists r \text{ s.t. } k_r^n = \kappa \text{ and } s_r^n \text{ is non-degenerate}\}$. From parts iii) and iv) of Prop. 2, $\lim_n \text{diameter}(U^n) = 0$. From assumption A10, $\lim_n \nu^n(U^n) = \nu^*(\lim_n U^n) = 0$. Moreover, since $K \subset \text{int}(A)$, both $\bar{U}^n$ and $\underline{U}^n$ must be nonempty for sufficiently large n, since otherwise, if say $\underline{U}^n = \emptyset$, then $t^n(\cdot) \approx \bar{a} > \sup(K)$, which would be superoptimal for all players. Hence, since strategies are monotone w.r.t. observable characteristics (Prop. 2), $\bar{k}^{\text{inf}} = \liminf(\bar{U}^n) \geq \underline{k}^{\text{sup}} = \limsup(\underline{U}^n)$. Moreover, since $\nu^*(\lim_n U^n) = 0$, $\exists k^*$ s.t. $t^*(\cdot) = k^* = \nu^*(\{k < \underline{k}^{\text{sup}}\})\underline{a} + \nu^*(\{k > \bar{k}^{\text{inf}}\})\bar{a}$. Necessarily, $k^* \in [\underline{k}^{\text{sup}}, \bar{k}^{\text{inf}}]$, since if $\underline{k}^{\text{sup}} > k^*$ there would for n sufficiently large exist r with $t^n(\cdot) \approx k^* < k_r^n \in \underline{U}^n$, so that r's payoff would increase by shifting from $\underline{a}$ to $\bar{a}$, a contradiction since $\mathbf{s}^n$ is an MPE. Finally, since $k^* \in [\underline{k}^{\text{sup}}, \bar{k}^{\text{inf}}]$ and $\nu^*((\underline{k}^{\text{sup}}, \bar{k}^{\text{inf}})) = 0$, $t^*(\cdot) = k^* = \nu^*(\{k < k^*\})\underline{a} + \nu^*(\{k > k^*\})\bar{a}$. ■